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Notes on Symplectic Geometry

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Contents

Chapter 1

Introduction

1.1 Newtonian mechanics

In newtonian mechanics the state of a mechanical system is described by a finite number of real parameters. The set of all possible positions, of a material point for example, is a finite dimensional smooth manifold M, called the *configuration space.* A motion of the system is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval. The velocity field of γ is smooth curve $\dot{\gamma} : I \to TM$. The (total space of the) tangent bundle TM of M is called the *phase space.*

According to Newton, the total force is a vector field F that acts on the points of the configuration space. Locally, a motion is a solution of the second order differential equation $F = m\ddot{x}$, where m is the mass. Equivalently, $\dot{\gamma}$ is locally a solution of the first order differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{1}{m}F(x) \end{pmatrix}.$$

This means that the motions are the projections on the configuration space M of the integral curves of a smooth vector field X on the phase space TM such that $\pi_* X = id$, where $\pi : TM \to M$ is the tangent bundle projection.

Definition 1.1. An (autonomous) newtonian mechamical system is a triple (M, g, X), where M is a smooth manifold, g is a Riemannian metric on M and X is a smooth vector field on TM such that $\pi_* X = id$. A motion of (M, g, X) is a smooth curve $\gamma : I \to M$ such that $\dot{\gamma} : I \to TM$ is an integral curve of X. The smmoth function $T : TM \to \mathbb{R}$ defined by $T(v) = \dfrac{1}{2}g(v, v)$ is called the kinetic energy.

Examples 1.2. (a) Let $M = \mathbb{R}$, so that we may identify TM with $\mathbb{R}^2$ and π with the projection onto the first coordinate. If g is the euclidean riemannian metric on $\mathbb{R}$ and

$$X = v\frac{\partial}{\partial x} + \frac{1}{m}(-k^2 x - \rho v)\frac{\partial}{\partial v}, \qquad k > 0, \quad \rho \geq 0,$$

then obviously $\pi_* X = id$ and a motion is a solution of the second order differential equation

$$m\ddot{x} = -k^2 x - \rho\dot{x}.$$

This mechanical system describes the oscillator.

(b) The geodesic vector field X of a Riemannian n-manifold (M, g) defines a newtonian mechanical system. Locally it has the expression

$$X = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} - \sum_{i,j,k=1}^{n} \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v^k}$$

where Γ_{ij}^k are the Christofell symbols.

Often a mechanical system has potential energy. This is a smooth function $V : M \to \mathbb{R}$. Let $\mathrm{grad}V$ be the gradient of V with respect to the Riemannian metric. If for every $v \in TM$ we set

$$\overline{\mathrm{grad}V} = \frac{d}{dt}\bigg|_{t=0} (v + t\mathrm{grad}V(\pi(v)),$$

then $\overline{\mathrm{grad}V} \in \mathcal{X}(TM)$ and $\pi_* \overline{\mathrm{grad}V} = 0$, since $\pi(v + t\mathrm{grad}V(\pi(v))) = \pi(v)$, for every $t \in \mathbb{R}$. Locally, if $g = (g_{ij})$ and $(g_{ij})^{-1} = (g^{ij})$, then

$$\mathrm{grad}V = \sum_{i,j=1}^{n} g^{ij} \frac{\partial V}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \text{and} \qquad \overline{\mathrm{grad}V} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial V}{\partial x^i} \frac{\partial}{\partial v^j}.$$

Definition 1.3. A newtonian mechanical system with potential energy is a triple (M, g, V), where (M, g) is a Riemannian manifold and $V : M \to \mathbb{R}$ is a smooth function called the potential energy.

The corresponding vector field on TM is $Y = X - \overline{\mathrm{grad}V}$, where X is the geodesic vector field. The smooth function $E = T + V \circ \pi : TM \to \mathbb{R}$ is called the mechanical energy.

Proposition 1.4. (Conservation of energy) *In a newtonian mechanical system with potential energy (M, g, V) the mechanical energy is a constant of motion.*

Proof. We want to show that $Y(E) = 0$. We compute locally, where we have

$$E = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij} v^i v^j + V \qquad \text{and}$$

$$Y = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} - \sum_{k=1}^{n} \Big(\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} \Big) \frac{\partial}{\partial v^k}.$$

Recall that

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \big(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \big).$$

We can now compute

$$Y(E)=\sum_{k=1}^n v^k\frac{\partial V}{\partial x^k}+\frac{1}{2}\sum_{i,j,k=1}^n\frac{\partial g_{ij}}{\partial x^k}v^iv^jv^k-\sum_{k=1}^n\Big(\sum_{i,j=1}^n\Gamma^k_{ij}v^iv^j+\sum_{i=1}^n\frac{\partial V}{\partial x^i}g^{ik}\Big)\frac{\partial V}{\partial v^k}$$

$$-\sum_{k=1}^n\Big(\sum_{i,j=1}^n\Gamma^k_{ij}v^iv^j+\sum_{i=1}^n\frac{\partial V}{\partial x^i}g^{ik}\Big)\Big(\sum_{i=1}^n g_{ik}v^i\Big)$$

$$=\sum_{k=1}^n v^k\frac{\partial V}{\partial x^k}-\Big(\sum_{i=1}^n\frac{\partial V}{\partial x^i}g^{ik}\Big)\Big(\sum_{i=1}^n g_{ik}v^i\Big)+\frac{1}{2}\sum_{i,j,k=1}^n\frac{\partial g_{ij}}{\partial x^k}v^iv^jv^k$$

$$-\sum_{k=1}^n\Big(\sum_{r=1}^n g_{rk}v^r\Big)\Big(\sum_{i,j=1}^n\frac{1}{2}\sum_{l=1}^n g^{kl}\big(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\big)v^iv^j\Big)$$

$$=\frac{1}{2}\sum_{i,j,k=1}^n\frac{\partial g_{ij}}{\partial x^k}v^iv^jv^k-\frac{1}{2}\sum_{i,j,l=1}^n\big(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\big)v^iv^jv^l=0. \qquad \square$$

A smooth curve $\gamma : I \to M$ is a motion of a newtonian mechanical system on M with potential energy V if and only if γ satisfies the second order differential equation

$$\nabla_{\dot\gamma}\dot\gamma=-\mathrm{grad}V$$

where ∇ is the Levi-Civita connection on M. In case V has an upper bound, then a motion is a geodesic with respect to a new Riemannian metric on M, possibly reparametrized. So, suppose that there exists some $e>0$ such that $V(x)<e$ for every $x\in M$. On M we consider the new Riemannian metric $g^*=(e-V)g$. Let γ be a montion with mechanical energy e, that is

$$\frac{1}{2}g(\dot\gamma(t),\dot\gamma(t))+V(\gamma(t))=e$$

for every $t\in I$. Since $e>V(\gamma(t))$, we have $\dot\gamma(t)\neq 0$ for every $t\in I$. The function $s:I\to\mathbb{R}$ with

$$s(t)=\sqrt{2}\int_{t_0}^t(e-V(\gamma(\tau))d\tau,$$

where $t_0\in I$, is smooth and strictly increasing. Let $\gamma^*=\gamma\circ s^{-1}$.

Theorem 1.5. (Jacobi-Maupertuis) *If γ is a motion of the mechanical system (M,g,V) and $V(x)<e$ for every $x\in M$, then its reparametrization γ^* is a geodesic with respect to the Riemannian metric $g^*=(e-V)g$.*

Proof. It suffices to carry out the computation locally. The Christofell symbols of the metric g^* are given by the formula

$$\Delta^k_{ij}=\Gamma^k_{ij}+\frac{1}{2(e-V)}\Big(-\frac{\partial V}{\partial x^i}\delta_{jk}-\frac{\partial V}{\partial x^j}\delta_{ik}+\sum_{l=1}^n\frac{\partial V}{\partial x^l}g^{lk}g_{ij}\Big).$$

If in the local coordinates we have $\gamma = (x^1, x^2, ..., x^n)$, then

$$\frac{dx^k}{ds} = \frac{dx^k}{dt} \cdot \frac{dt}{ds} = \frac{1}{\sqrt{2}(e-V)} \cdot \frac{dx^k}{dt}$$

and so

$$\frac{d^2x^k}{ds^2} = \frac{1}{2(e-V)^2} \cdot \frac{d^2x^k}{d^2t} + \frac{1}{2(e-V)^3} \cdot \frac{dx^k}{dt} \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} \frac{dx^l}{dt}.$$

Since

$$\frac{d^2x^k}{d^2t} = -\sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} - \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{kl}$$

and

$$\sum_{i,j=1}^{n} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 2(e-V)$$

substituting we get

$$\frac{d^2x^k}{ds^2} + \sum_{i,j=1}^{n} \Delta_{ij}^k \frac{dx^i}{ds} \frac{dx^j}{ds} =$$

$$-\frac{1}{2(e-V)^2} \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} - \frac{1}{2(e-V)^2} \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{kl} + \frac{1}{2(e-V)^3} \frac{dx^k}{dt} \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} \frac{dx^l}{dt}$$

$$+\frac{1}{2(e-V)^2} \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} - \frac{1}{4(e-V)^3} \frac{dx^k}{dt} \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} \frac{dx^i}{dt} - \frac{1}{4(e-V)^3} \frac{dx^k}{dt} \sum_{j=1}^{n} \frac{\partial V}{\partial x^j} \frac{dx^j}{dt}$$

$$+\frac{1}{4(e-V)^3} \left(\sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{kl} \right) \left(\sum_{i,j=1}^{n} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) =$$

$$-\frac{1}{2(e-V)^2} \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{kl} + \frac{1}{4(e-V)^3} \left(\sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{kl} \right) \left(\sum_{i,j=1}^{n} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) = 0. \qquad \square$$

1.2 Lagrangian mechanics

Let (M, g, V) be a newtonian mechanical system with potential energy V and let $L : TM \to \mathbb{R}$ be the smooth function $L = T - V \circ \pi$, where T is the kinetic energy and $\pi : TM \to M$ is the tangent bundle projection.

Theorem 2.1. (d'Alembert-Lagrange) *A smooth curve $\gamma : I \to M$ is a motion of the mechanical system (M, g, V) if and only if*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot\gamma(t))\right) = \frac{\partial L}{\partial x^i}(\dot\gamma(t))$$

for every $t \in I$ and $i = 1, 2..., n$, where n is the dimension of M.

Proof. Suppose that in local coordinates we have $\gamma = (x^1, x^2, ..., x^n)$. Recall that γ is a motion of (M, g, V) if and only if

$$\ddot{x}^k = -\sum_{i,j=1}^{n} \Gamma^k_{ij}\dot{x}^i\dot{x}^j - \sum_{l=1}^{n} \frac{\partial V}{\partial x^l} g^{lk}.$$

Since

$$L(\dot{\gamma}) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}\dot{x}^i\dot{x}^j - V(\gamma),$$

for every $i = 1, 2, ..., n$ we have

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v^i}(\dot{\gamma}(t))\Big) - \frac{\partial L}{\partial x^i}(\dot{\gamma}(t)) = \frac{d}{dt}\left(\sum_{j=1}^{n} g_{ij}\dot{x}^j\right) - \frac{1}{2}\sum_{m,l=1}^{n}\frac{\partial g_{ml}}{\partial x^i}\dot{x}^m\dot{x}^l + \frac{\partial V}{\partial x^i}(\gamma(t)) =$$

$$\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial g_{ij}}{\partial x^l}\dot{x}^l\dot{x}^j + \sum_{j=1}^{n} g_{ij}\ddot{x}^j - \frac{1}{2}\sum_{m,l=1}^{n}\frac{\partial g_{ml}}{\partial x^i}\dot{x}^m\dot{x}^l + \frac{\partial V}{\partial x^i}(\gamma(t)) =$$

$$\sum_{m,l=1}^{n}\left(\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}\right)\dot{x}^m\dot{x}^l + \sum_{j=1}^{n} g_{ij}\ddot{x}^j + \frac{\partial V}{\partial x^i}(\gamma(t)).$$

Taking the image of the vector with these coordinates by $(g_{ij})^{-1} = (g^{ij})$, we see that the equations in the statement of the theorem are equivalent to

$$0 = \sum_{i=1}^{n} g^{ik}\left(\sum_{m,l=1}^{n}\left(\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}\right)\dot{x}^m\dot{x}^l\right) + \sum_{j=1}^{n} g^{ik} g_{ij}\ddot{x}^j + \sum_{i=1}^{n}\frac{\partial V}{\partial x^i}g^{ik} =$$

$$\ddot{x}^k + \sum_{m,l=1}^{n} g^{ik}\Big(\sum_{i=1}^{n}\left(\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}\right)\dot{x}^m\dot{x}^l + \sum_{i=1}^{n}\frac{\partial V}{\partial x^i}g^{ik} =$$

$$\ddot{x}^k + \sum_{m,l=1}^{n}\Gamma^k_{ml}\dot{x}^m\dot{x}^l + \sum_{i=1}^{n}\frac{\partial V}{\partial x^i}g^{ik}. \qquad \square$$

Generalizing we give the following definition.

Definition 2.2. An autonomous Lagrangian system is a couple (M, L), where M is a smooth manifold and $L : TM \to \mathbb{R}$ is a smooth function, called the Lagrangian. A Lagrangian motion is a smooth curve $\gamma : I \to M$ which locally satisfies the system of differential equations

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v^i}(\dot{\gamma}(t))\Big) = \frac{\partial L}{\partial x^i}(\dot{\gamma}(t))$$

for $t \in I$ and $i = 1, 2..., n$, where n is the dimension of M. These equations are called the Euler-Lagrange equations.

Example 2.3. Let (M, g) be a pseudo-Riemannian n-manifold and A be a smooth 1-form on M. The Lagrangian

$$L(v) = \frac{1}{2}mg(v, v) - A(v)$$

generalizes the motion of a charged particle of mass m under the influence of an electromagnetic field defined by the 1-form A. Let $(U, x^1, x^2, ..., x^n)$ be a local system of coordinates on M and $(\pi^{-1}(U), x^1, x^2, ..., x^n, v^1, v^2, ..., v^n)$ be the corresponding local system of coordinates on TM. In local coordinates L is given by the formula

$$L(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n) = \frac{1}{2} m \sum_{i,j=1}^{n} g_{ij} v^i v^j - \sum_{i=1}^{n} A_i dx^i,$$

where (g_{ij}) is the matrix of the pseudo-Riemannian metric g and $A = \sum_{i=1}^{n} A_i dx^i$ on U. A smooth curve $\gamma : I \to M$ is a Lagrange motion if and only if it satisfies the Euler-Lagrange equations. In our case the right hand side of the Euler-Lagrange equations is

$$\frac{\partial L}{\partial x^i}(\dot\gamma(t)) = \frac{1}{2} m \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{dt} \frac{dx^j}{dt} - \sum_{i=1}^{n} \frac{\partial A_i}{\partial x^k} \frac{dx^i}{dt},$$

and the left hand side

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot\gamma(t))\right) = m \sum_{i,j=1}^{n} \frac{\partial g_{ik}}{\partial x^j} \frac{dx^j}{dt} \frac{dx^i}{dt} + m \sum_{i=1}^{n} g_{ik} \frac{d^2 x^i}{dt^2} - \sum_{i=1}^{n} \frac{\partial A_k}{\partial x^i} \frac{dx^i}{dt}.$$

So the Euler-Lagrange equations are equivalent to

$$\sum_{i=1}^{n} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) \frac{dx^i}{dt} = m \sum_{i=1}^{n} g_{ik} \frac{d^2 x^i}{dt^2} + m \sum_{i,j=1}^{n} \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k}\right) \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

On the other hand

$$dA(\dot\gamma(t), \frac{\partial}{\partial x^k}) = \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i(\dot\gamma(t), \frac{\partial}{\partial x^k}) = \sum_{i=1}^{n} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) \frac{dx^i}{dt}.$$

Recall that the covariant derivative formula along γ gives

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{l=1}^{n} \frac{d^2 x^l}{dt^2} \frac{\partial}{\partial x^l} + \sum_{i,j,l=1}^{n} \Gamma_{ij}^l \frac{dx^i}{dt} \frac{dx^j}{dt} \frac{\partial}{\partial x^l}$$

and so

$$g(m\nabla_{\dot\gamma}\dot\gamma, \frac{\partial}{\partial x^k}) = m \sum_{l=1}^{n} g_{lk} \frac{d^2 x^l}{dt^2} + m \sum_{i,j,l=1}^{n} g_{lk} \Gamma_{ij}^l \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

Since

$$\sum_{l=1}^{n} g_{lk} \Gamma_{ij}^l = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

we get

$$\sum_{i,j,l=1}^{n} g_{lk} \Gamma_{ij}^l \frac{dx^i}{dt} \frac{dx^j}{dt} = \sum_{i,j=1}^{n} \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k}\right) \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

We conclude now that the Euler-Lagrange equations have the form

$$g(m\nabla_{\dot\gamma}\dot\gamma, \frac{\partial}{\partial x^k}) = dA(\dot\gamma, \frac{\partial}{\partial x^k}), \qquad k = 1, 2, ..., n$$

or independently of local coordinates

$$m\nabla_{\dot\gamma}\dot\gamma = \mathrm{grad}(i_{\dot\gamma}(dA)),$$

where the gradient is taken with respect to the pseudo-Riemannian metric g.

Theorem 2.4. (Least Action Principle) *Let (M, L) be a Lagrangian system. A smooth curve $\gamma : [a, b] \to M$ is a Lagrangian motion if and only if for every smooth variation $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ of γ with fixed endpoints, so that $\Gamma(0, t) = \gamma(t)$ for $a \le t \le b$, we have*

$$\frac{\partial}{\partial s}\bigg|_{s=0}\int_a^b L(\frac{\partial\Gamma}{\partial t}(s,t))dt = 0.$$

Proof. It suffices to carry out the computations locally. We have

$$\frac{\partial}{\partial s}\bigg|_{s=0}\int_a^b L(\frac{\partial\Gamma}{\partial t}(s,t))dt = \int_a^b \frac{\partial}{\partial s}\bigg|_{s=0} L(\frac{\partial\Gamma}{\partial t}(s,t))dt =$$

$$\int_a^b \left[\sum_{i=1}^n \frac{\partial L}{\partial x^i}(\dot\gamma(t))\frac{\partial\Gamma}{\partial s}(0,t) + \sum_{i=1}^n \frac{\partial L}{\partial v^i}(\dot\gamma(t))\frac{\partial}{\partial s}\bigg|_{s=0}(\frac{\partial\Gamma}{\partial t}(s,t))\right]dt =$$

$$\int_a^b \left[\sum_{i=1}^n \frac{\partial L}{\partial x^i}(\dot\gamma(t))\frac{\partial\Gamma}{\partial s}(0,t) + \sum_{i=1}^n \frac{\partial L}{\partial v^i}(\dot\gamma(t))\frac{\partial}{\partial t}(\frac{\partial\Gamma}{\partial s}(0,t))\right]dt =$$

$$\int_a^b \left[\sum_{i=1}^n \frac{\partial L}{\partial x^i}(\dot\gamma(t))\frac{\partial\Gamma}{\partial s}(0,t) + \frac{d}{dt}\left(\sum_{i=1}^n \frac{\partial L}{\partial v^i}(\dot\gamma(t))\frac{\partial\Gamma}{\partial s}(0,t)\right) - \sum_{i=1}^n \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot\gamma(t))\right)\frac{\partial\Gamma}{\partial s}(0,t)\right]dt =$$

$$\int_a^b \sum_{i=1}^n \left[\frac{\partial L}{\partial x^i}(\dot\gamma(t)) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot\gamma(t))\right)\right]\frac{\partial\Gamma}{\partial s}(0,t)dt,$$

because $\frac{\partial\Gamma}{\partial s}(0,a) = \frac{\partial\Gamma}{\partial s}(0,b) = 0$ since the variation is with fixed endpoints. This means that

$$\frac{\partial}{\partial s}\bigg|_{s=0}\int_a^b L(\frac{\partial\Gamma}{\partial t}(s,t))dt = 0$$

if and only if

$$\frac{d}{dt}(\frac{\partial L}{\partial v^i}(\dot\gamma(t))) = \frac{\partial L}{\partial x^i}(\dot\gamma(t))$$

for $i = 1$, 2..., n, because $\frac{\partial\Gamma}{\partial s}(0,t)$ can take any value. □

As in newtonian mechanical systems with potential energy, one can define the notion of mechanical energy for Lagrangian systems also. In order to do this, we shall need to define first the Legendre transformation. So let $L : TM \to \mathbb{R}$ be a Lagrangian and $p \in M$, $v \in T_pM$. The derivative

$$(L|_{T_pM})_{*v} : T_v(T_pM) \cong T_pM \to \mathbb{R}$$

can be considered as an element of the dual tangent space T_p^*M.

Definition 2.5. The Legendre transformation of a Lagrangian system (M, L) is the map $\mathcal{L} : TM \to T^*M$ defined by $\mathcal{L}(p, v) = (L|_{T_pM})_{*v}$.

Example 2.6. If $L = \frac{1}{2}g - V$ is the Lagrangian of a newtonian mechanical system with potential energy (M, g, V), then for every $p \in M$ and v, $w \in T_pM$ we have $\mathcal{L}(v)(w) = g(v, w)$. Thus, in this case the Legendre transformation $\mathcal{L} : TM \to T^*M$ is the natural isomorphism defined by the Riemannian metric.

Definition 2.7. The energy of a Lagrangian system (M, L) is the smooth function $E : TM \to \mathbb{R}$ defined by $E(v) = \mathcal{L}(v)(v) - L(v)$.

If $(x^1, x^2, ..., x^n)$ is a system of local coordinates on M with corresponding local coordinates $(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n)$ on TM, then

$$E(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} v^i - L(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n).$$

In the case of a newtonian mechanical system with potential energy (M, g, V) the above definition gives

$$E(v) = \mathcal{L}(v)(v) - L(v) = g(v, v) - \frac{1}{2}g(v, v) + V(\pi(v)) = \frac{1}{2}g(v, v) + V(\pi(v)),$$

which coincides with the previous definition.

Example 2.8. We shall compute the Legendre transformation and the energy of the Lagrangian system of example 2.3 using the same notation. Considering local coordinates $(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n)$ on TM, we have

$$(L|_{T_pM})(v^1, v^2, ..., v^n) = \frac{1}{2}m \sum_{i,j=1}^{n} g_{ij} v^i v^j - \sum_{i=1}^{n} A_i v^i.$$

Differentiating we get

$$(L|_{T_pM})_{*v} = m \sum_{i,j=1}^{n} g_{ij} v^i dv^j - \sum_{i=1}^{n} A_i dv^i.$$

We conclude now that

$$\mathcal{L}(v)(w) = (L|_{T_pM})_{*v}(w) = mg(v, w) - A(w).$$

The energy here is

$$E(v) = \mathcal{L}(v)(v) - L(v) = \frac{1}{2}mg(v, v),$$

and so does not depend on the 1-form A, which represents the magnetic field. This reflects the fact that the magnetic field does not produce work.

Theorem 2.9. (Conservation of energy) *In a Lagrangian system the energy is a constant of motion.*

Proof. Considering local coordinates on M, let $\gamma = (x^1, x^2, ..., x^n)$ be a Lagrangian motion. Then

$$E(\dot{\gamma}(t)) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i}\frac{dx^i}{dt} - L(\dot{\gamma}(t))$$

and differentiating

$$\frac{d}{dt}(E(\dot{\gamma}(t))) = \sum_{i,j=1}^{n}\left(\frac{\partial^2 L}{\partial v^i \partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j}\frac{dx^i}{dt}\frac{d^2x^j}{dt^2}\right) + \sum_{i=1}^{n}\frac{\partial L}{\partial v^i}\frac{d^2x^i}{dt^2}$$

$$-\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \sum_{i=1}^{n}\frac{\partial L}{\partial v^i}\frac{d^2x^i}{dt^2} =$$

$$\sum_{i,j=1}^{n}\left(\frac{\partial^2 L}{\partial v^i \partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j}\frac{dx^i}{dt}\frac{d^2x^j}{dt^2}\right) - \sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\frac{dx^i}{dt}.$$

But from the Euler-Lagrange equations we have

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}(\frac{\partial L}{\partial v^i}) = \sum_{j=1}^{n}\left(\frac{\partial^2 L}{\partial v^i \partial x^j}\frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j}\frac{d^2x^j}{dt^2}\right)$$

and so substituting we get $\frac{d}{dt}(E(\dot{\gamma}(t))) = 0$. □

Apart from the energy, one can have constants of motion from symmetries of the Lagrangian.

Theorem 2.10. (Noether) *Let (M, L) be a Lagrangian system and X a complete smooth vector field on M with flow $(\phi_t)_{t\in\mathbb{R}}$. If $L((\phi_t)_{*p}(v)) = L(v)$ for every $v \in T_pM$, $p \in M$ and $t \in \mathbb{R}$, then the smooth function $f_X : TM \to \mathbb{R}$ defined by*

$$f_X(v) = \lim_{s\to 0}\frac{L(v + sX(\pi(v))) - L(v)}{s}$$

is a constant of motion.

Proof. Considering local coordinates, let $\phi_t = (\phi_t^1, \phi_t^2, ..., \phi_t^n)$. Since L is $(\phi_t)_*$ invariant, if $\gamma = (x^1, x^2, ..., x^n)$ is a Lagrangian motion, differentiating the equation $L((\phi_s)_{*\gamma(t)}(\dot{\gamma}(t))) = L(\dot{\gamma}(t))$ with respect to s, we have

$$\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\left(\frac{\partial \phi_s^i}{\partial s}\right)_{s=0} + \sum_{i,j=1}^{n}\frac{\partial L}{\partial v^i}\left(\frac{\partial^2 \phi_s^i}{\partial x^j \partial s}\right)_{s=0}\frac{dx^j}{dt} = 0.$$

Since $f_X(v)$ is the directional derivative of $L|_{T_{\pi(v)}M}$ in the direction of $X(\pi(v))$ and

$$X = \sum_{i=1}^{n}\left(\frac{\partial \phi_t^i}{\partial t}\right)_{t=0}\frac{\partial}{\partial x^i},$$

we have

$$f_X(\dot\gamma(t)) = \sum_{i=1}^n \frac{\partial L}{\partial v^i}\left(\frac{\partial \phi_s^i}{\partial s}\right)_{s=0}.$$

Using now the Euler-Lagrange equations we compute

$$\frac{d}{dt}(f_X(\dot\gamma(t))) = \sum_{i=1}^n \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\left(\frac{\partial \phi_s^i}{\partial s}\right)_{s=0} + \sum_{i=1}^n \frac{\partial L}{\partial v^i}\frac{d}{dt}\left(\frac{\partial \phi_s^i}{\partial s}\right)_{s=0} =$$

$$\sum_{i=1}^n \frac{\partial L}{\partial x^i}\left(\frac{\partial \phi_s^i}{\partial s}\right)_{s=0} + \sum_{i,j=1}^n \frac{\partial L}{\partial v^i}\left(\frac{\partial^2 \phi_s^i}{\partial x^j \partial s}\right)_{s=0}\frac{dx^j}{dt} = 0. \qquad \square$$

Examples 2.11. (a) Let (M, g, V) be a newtonian mechanical system with potential energy and X be a complete vector field, which is a symmetry of the system. Then $f_X(v) = g(v, X(\pi(v)))$. The restriction to a fiber of the tangent bundle of f_X is linear in this case.

(b) Let X be a complete vector field which we assume to be a symmetry of the Lagrangian system of example 2.3. For instance, this is the case if the flow of X preserves the pseudo-Riemannian metric on M and the 1-form A. Then the Lagrangian is X-invariant and the first integral provided form Noether's theorem is $f_X(v) = mg(v, X) - A(X)$.

1.3 The equations of Hamilton

A Lagrangian system (M, L) is called hyperregular if the Legendre transformation $\mathcal{L} : TM \to T^*M$ is a diffeomorphism. For example a newtonian mechanical system with potential energy and the system of example 2.3 are hyperregular.

Definition 3.1. In a hyperregular Lagrangian system as above, the smooth function $H = E \circ \mathcal{L}^{-1} : T^*M \to \mathbb{R}$, where E is the energy, is called the Hamiltonian function of the system.

Example 3.2. Let (M, g, V) is a newtonian mechanical system with potential energy. The Legendre transformation gives

$$q^i = x^i, \quad p_i = \frac{\partial L}{\partial v^i} = \sum_{j=1}^n g_{ij} v^j.$$

The inverse Legendre transformation is given by

$$x^i = q^i, \quad v^i = \sum_{j=1}^n g^{ij} p_j.$$

So we have

$$E = \frac{1}{2}\sum_{i,j=1}^n g_{ij} v^i v^j + V(x^1, x^2, ..., x^n),$$

$$L = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}v^i v^j - V(x^1, x^2, ..., x^n)$$

and therefore

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}p_i p_j + V(q^1, q^2, ..., q^n).$$

Theorem 3.3. (Hamilton) *Let (M, L) be a hyperregular Lagrangian system on the n-dimensional manifold M. A smooth curve $\gamma : I \to M$ is a Lagrangian motion if and only if the smooth curve $\mathcal{L}\circ\dot{\gamma} : I \to T^*M$ locally solves the system of differential equations*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, ..., n.$$

Proof. In the local coordinates $(x^1, x^2, ..., x^n)$ of a chart on M the Legendre transformation is given by the formulas

$$q^i = x^i, \quad p_i = \frac{\partial L}{\partial v^i}, \quad i = 1, 2, ..., n,$$

where $(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n)$ are the local coordinates of the corresponding chart on TM. Inversing,

$$x^i = q^i, \quad v^i = y^i(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n), \quad i = 1, 2, ..., n$$

for some smooth functions y^1, y^2,...,y^n. From the definitions of the energy E and the Hamiltonian H, we have

$$H = E \circ \mathcal{L}^{-1} = \sum_{j=1}^{n} p_j y^j - L(q^1, q^2, ..., q^n, y^1, y^2, ..., y^n)$$

and differentiating the chain rule gives

$$\frac{\partial H}{\partial p_i} = y^i + \sum_{j=1}^{n} p_j \frac{\partial y^j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial v^j}\frac{\partial y^j}{\partial p_i} = y^i$$

$$\frac{\partial H}{\partial q^i} = \sum_{j=1}^{n} p_j \frac{\partial y^j}{\partial q^i} - \frac{\partial L}{\partial x^i} - \sum_{j=1}^{n} \frac{\partial L}{\partial v^j}\frac{\partial y^j}{\partial q^i} = -\frac{\partial L}{\partial x^i}.$$

If $\gamma(t) = (x^1(t), x^(t), ..., x^n(t))$ is a smooth curve in local coordinates on M, then

$$\mathcal{L}(\dot{\gamma}(t)) = (x^1(t), x^(t), ..., x^n(t), \frac{\partial L}{\partial v^1}(\dot{\gamma}(t)), \frac{\partial L}{\partial v^2}(\dot{\gamma}(t)), ..., \frac{\partial L}{\partial v^n}(\dot{\gamma}(t))).$$

Now γ is a Lagrangian motion if and only if

$$\dot{x}^i = v^i \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot{\gamma})\right) = \frac{\partial L}{\partial x^i}(\dot{\gamma})$$

or equivalently

$$\dot{q}^i = \dot{x}^i = v^i = y^i = \frac{\partial H}{\partial p_i} \quad \text{and}$$

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot{\gamma})\right) = \frac{\partial L}{\partial x^i}(\dot{\gamma}) = -\frac{\partial H}{\partial q^i}. \qquad \square$$

Chapter 2

Symplectic spaces

2.1 Symplectic vector spaces

A *synplectic form* on a (real) vector space V of finite dimension is a non-degenerate, skew-symmetric, bilinear form $\omega : V \times V \to \mathbb{R}$. This means that the map $\tilde{\omega} : V \to V^*$ defined by $\tilde{\omega}(v)(w) = \omega(v, w)$, for v, $w \in V$, is a linear isomorphism. The pair (V, ω) is then called a *symplectic vector space*.

Lemma 1.1. (Cartan) *Let V be a vector space of dimension n and ω be a skew-symmetric, bilinear form on V. If $\omega \neq 0$, then the rank of $\tilde{\omega}$ is even. If* $\dim \tilde{\omega}(V) = 2k$*, there exists a basis l^1, l^2,...,l^{2k} of $\tilde{\omega}(V)$ such that*

$$\omega = \sum_{j=1}^{k} l^{2j-1} \wedge l^{2j}.$$

Proof. Let $\{v_1, v_2, ..., v_n\}$ be a basis of V and $\{v_1^*, v_2^*, ..., v_n^*\}$ be the corresponding dual basis of V^*. If $a_{ij} = \omega(v_i, v_j)$, $i < j$, then

$$\omega = \sum_{i<j} a_{ij} v_i^* \wedge v_j^*.$$

Since $\omega \neq 0$, there are some $1 \leq i < j \leq n$ such that $a_{ij} \neq 0$. We may assume that $a_{12} \neq 0$, changing the numbering if necessary. Let

$$l^1 = \frac{1}{a_{12}} \tilde{\omega}(v_1) = v_2^* + \frac{1}{a_{12}} \sum_{j=3}^{n} a_{1j} v_j^*,$$

$$l^2 = \tilde{\omega}(v_2) = -a_{12} v_1^* + \sum_{j=3}^{n} a_{2j} v_j^*.$$

The set $\{l^1, l^2, v_3^*, ..., v_n^*\}$ is now a new basis of V^*. If $\omega_1 = \omega - l^1 \wedge l^2$, then

$$\tilde{\omega}_1(v_1) = a_{12} l^1 - l^1(v_1) l^2 + l^2(v_1) l^1 = a_{12} l^1 - 0 - a_{12} l^1 = 0,$$

$$\tilde{\omega}_1(v_2) = l^2 - l^1(v_2)l^2 + l^2(v_2)l^1 = l^2 - l^2 + 0 = 0.$$

Thus, ω_1 is an element of the subalgebra of the exterior algebra of V generated by v_3^*,,...,v_n^*. If $\omega_1 = 0$, then $\omega = l^1 \wedge l^2$. If $\omega_1 \neq 0$, we repeat the above taking ω_1 in the place of ω. So, inductively, we arrive at the conclusion, since V has finite dimension. $\square$

Note that as the proof of Cartan's lemma shows, l^2 can be chosen to be any non-zero element of $\tilde{\omega}(V)$.

Corollary 1.2. *If ω is a skew-symmetric, bilinear form of rank $2k$ of a vector space, then k is the maximal positive integer such that $\omega \wedge ... \wedge \omega \neq 0$ (k times).*

Proof. Indeed from Cartan's lemma, the $(k+1)$-fold wedge product of ω with itself is equal to 0 and the k-fold is $\omega \wedge ... \wedge \omega = k \cdot l^1 \wedge ... \wedge l^{2k} \neq 0$. $\square$

By Cartan's lemma, if (V, ω) is a symplectic vector space of finite dimension, there exists some $n \in \mathbb{N}$ such that $\dim V = 2n$ and there exists a basis $\{l^1, l^2, ..., l^{2n}\}$ of V^* such that

$$\omega = \sum_{j=1}^{n} l^j \wedge l^{n+j}.$$

If $W \leq V$, we set $W^\perp = \{v \in V : \omega(w, v) = 0 \quad \text{for every} \quad w \in W\}$. Obviously, $W^\perp \leq V$ and $\tilde{\omega}(W^\perp) = \{a \in V^* : a|_W = 0\}$, since $\tilde{\omega}$ is an isomorphism.

Lemma 1.3. *Let (V, ω) be a symplectic vector space of dimension $2n$ and W_1, W_2, W be subspaces of V. Then the following hold:*

(a) $\dim W + \dim W^\perp = \dim V = 2n$.
(b) $W^{\perp\perp} = W$.
(c) $W_1 \leq W_2$ *if and only if* $W_2^\perp \leq W_1^\perp$.
(d) $W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$.

Proof. (a) Let $W^o = \tilde{\omega}(W^\perp)$. Since $\tilde{\omega}$ is an isomorphism, it suffices to show that $\dim W^o = 2n - k$, where $k = \dim W$. Let $\{w_1, ..., w_{2n}\}$ be a basis of V such that $\{w_1, ..., w_k\}$ is a basis of W. Let $\{w_1^*, ..., w_{2n}^*\}$ be the correspomding dual basis of V^*. If now $a \in W^o \leq V^*$, there are a_1,...,$a_{2n} \in \mathbb{R}$ such that

$$a = \sum_{i=1}^{2n} a_i w_i^*$$

and then

$$a_j = \sum_{i=1}^{2n} a_i w_i^*(w_j) = a(w_j) = 0$$

for every $1 \leq j \leq k$. This means that

$$a = \sum_{i=k+1}^{2n} a_i w_i^*$$

and shows that $\{w^*_{k+1}, ..., w^*_{2n}\}$ generates W^o. Since $\{w^*_{k+1}, ..., w^*_{2n}\}$ is also a linearly independent set, it is a basis of W^o and hence $\dim W^o = 2n - k$.

(b) Evidently, $W \leq W^{\perp\perp}$ and since $\dim W = \dim W^{\perp\perp}$, by (a), we have $W = W^{\perp\perp}$.

(c) If $W_1 \leq W_2$, then obviously $W_2^\perp \leq W_1^\perp$. Conversely, if $W_2^\perp \leq W_1^\perp$, then from (b) we have $W_1 = W_1^{\perp\perp} \leq W_2^{\perp\perp} = W_2$.

(d) From (c) we have $(W_1 + W_2)^\perp \leq W_1^\perp \cap W_2^\perp$. On the other hand,

$$\dim V = \dim W_1^\perp + \dim W_2^\perp - \dim(W_1^\perp \cap W_2^\perp)$$

$$= \dim V - \dim W_1 + \dim V - \dim W_2 - \dim(W_1^\perp \cap W_2^\perp)$$

from (a), and so

$$\dim V = \dim W_1 + \dim W_2 + \dim(W_1^\perp \cap W_2^\perp).$$

But $\dim V = \dim(W_1 + W_2) + \dim(W_1 + W_2)^\perp$, again from (a), and hence

$$\dim V \leq \dim W_1 + \dim W_2 + \dim(W_1 + W_2)^\perp.$$

It follows that $\dim(W_1^\perp \cap W_2^\perp) \leq \dim(W_1 + W_2)^\perp$, which means that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$. $\square$

Example 1.4. Let W be a vector space of dimension n. On $W \times W^*$ consider the skew-symmetric, bilinear form ω defined by

$$\omega((w, a), (w', a')) = a'(w) - a(w').$$

If now $\tilde{\omega}(w, a) = 0$, then $0 = \omega((w, a), (w', 0)) = -a(w')$ for every $w' \in W$. Thus $a = 0$. Similarly, $0 = \omega((w, a), (0, a')) = a'(w)$ for every $a' \in W^*$. Hence $w = 0$. This shows that $(W \times W^*, \omega)$ is a symplectic vector space.

Let (V, ω) be a symplectic vector space of dimension $2n$. A subspace $W \leq V$ is called

(i) isotropic, if $W \leq W^\perp$, and then $\dim W \leq n$,

(ii) coisotropic, if $W^\perp \leq W$, and then $\dim W \geq n$,

(iii) Lagrangian, if $W = W^\perp$, and then $\dim W = n$, and

(iv) symplectic, if $W \cap W^\perp = \{0\}$, and then $\dim W$ is even.

For instance, in Example 1.4, the subspaces $W \times \{0\}$ and $\{0\} \times W^*$ are Lagrangian.

Proposition 1.5. *Every isotropic subspace of a symplectic vector space (V, ω) is contained in a Lagrangian subspace.*

Proof. Let $W \leq V$ be an isotropic subspace, which is not Lagrangian itself. There exists $v \in W^\perp \setminus W$. Then $< v >$ is an isotropic subspace of V and therefore $< v > \leq < v >^\perp \cap W^\perp$. By Lemma 1.3(c), $W \leq < v >^\perp$, and thus $W \leq < v >^\perp \cap W^\perp$, since W is isotropic. From Lemma 1.3(d) we have now

$$< v > + W \leq < v >^\perp \cap W^\perp = (< v > + W)^\perp,$$

which means that $< v > +W$ is isotropic. Since $\dim V$ is finite, repeating this process we arrive after a finite number of steps at a Lagrangian subspace. $\square$

On $\mathbb{R}^{2n} \cong \mathbb{R}^n \times (\mathbb{R}^n)^*$ we consider the canonical symplectic structure ω of Example 1.4. Let $\langle , \rangle$ be the euclidean inner product on $\mathbb{R}^{2n}$ and $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the orthogonal transformation $J(x, y) = (-y, x)$ for x, $y \in \mathbb{R}^n$. Then, $J^2 = -id$ and

$$\omega((x, y), (x', y')) = \langle J(x, y), (x', y')\rangle.$$

If V is any symplectic vector space of dimension $2n$, there exists a basis $\{l^1, l^2, ..., l^{2n}\}$ of V such that the symplectic form of V is

$$\sum_{j=1}^{n} l^j \wedge l^{n+j}.$$

If now $\{e_1, e_2, ..., e_{2n}\}$ is the canonical basis of $\mathbb{R}^{2n}$, then the linear map sending l^j to e_j is an isomorphism pulling the canonical symplectic structure ω of Example 1.4 back to $\sum_{j=1}^{n} l^j \wedge l^{n+j}$. This means that there exists only one symplectic vector space of dimension $2n$ up to symplectic isomorphism.

A linear map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *symplectic* if $f^*\omega = \omega$. If A is the matrix of f with respect to the canonical basis, then

$$\langle Jv, w\rangle = \omega(v, w) = \omega(Av, Aw) = \langle JAv, Aw\rangle = \langle A^tJAv, w\rangle,$$

for every v, $w \in \mathbb{R}^{2n}$. So, f is symplectic if and only if $A^tJA = J$. In particular, $\det A \neq 0$ and f is a linear isomorphism. The set symplectic linear maps

$$\mathrm{Sp}(n, \mathbb{R}) = \{A \in \mathbb{R}^{2n\times 2n} : A^tJA = J\}$$

is now a Lie group and is called the symplectic group. To see that $\mathrm{Sp}(n, \mathbb{R})$ is a Lie group, let $F : GL(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R})$ be the smooth map

$$F(A) = A^tJA.$$

Then, $\mathrm{Sp}(n, \mathbb{R}) = F^{-1}(J)$ and it suffices to show that J is a regular value of F. The derivative of F at A is $F_{*A}(H) = H^tJA + A^tJH$, $H \in \mathbb{R}^{2n\times 2n}$. Let $A \in \mathrm{Sp}(n, \mathbb{R})$ and $B \in \mathfrak{so}(2n, \mathbb{R})$. If $H = -\dfrac{1}{2}AJB$, since $A^tJ = JA^{-1}$, then

$$H^tJA + A^tJH = -\frac{1}{2}(JB)^tJ + J(-\frac{1}{2}JB) = -\frac{1}{2}B^tJ^tJ - \frac{1}{2}J^2B = B.$$

This shows that F_{*A} is a linear epimorphism. Hence $\mathrm{Sp}(n, \mathbb{R})$ is a Lie group with Lie albegra

$$\mathfrak{sp}(n, \mathbb{R}) = \{H \in \mathbb{R}^{2n\times 2n} : H^tJ + JH = 0\}.$$

and has dimension $2n^2 + n$.

Let $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the usual hermitian product defined by

$$h(u, v) = \langle u, v\rangle + i\omega(u, v).$$

Then, $h(u, Jv) = \langle u, Jv\rangle + i\langle Ju, Jv\rangle = \langle J^t u, v\rangle + i\langle u, v\rangle = i\langle u, v\rangle - \langle Ju, v\rangle = ih(u, v)$. If now $A \in GL(2n, \mathbb{R})$, then identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ we have

$A \in U(n)$ if and only if $h(Au, Av) = h(u, v)$ for every u, $v \in \mathbb{R}^{2n}$
if and only if $\langle Au, Av\rangle = \langle u, v\rangle$ and $\omega(Au, Av) = \omega(u, v)$ for every u, $v \in \mathbb{R}^{2n}$
if and only if $A \in O(2n, \mathbb{R}) \cap \mathrm{Sp}(n, \mathbb{R})$
if and only if $JA = AJ$ and $\langle Au, Av\rangle = \langle u, v\rangle$ for every u, $v \in \mathbb{R}^{2n}$
if and only if $A \in O(2n, \mathbb{R}) \cap GL(n, \mathbb{C})$
if and only if $\langle AJu, Av\rangle = \langle JAu, Av\rangle = \langle Ju, v\rangle$ for every u, $v \in \mathbb{R}^{2n}$
if and only if $A \in \mathrm{Sp}(n, \mathbb{R}) \cap GL(n, \mathbb{C})$.

In other words,

$$O(2n, \mathbb{R}) \cap \mathrm{Sp}(n, \mathbb{R}) = O(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) = \mathrm{Sp}(n, \mathbb{R}) \cap GL(n, \mathbb{C}) = U(n).$$

There is a symplectic version of the well known polar decomposition for the general linear groups. If $A \in \mathrm{Sp}(n, \mathbb{R})$, then

$$(A^t)^t JA^t = AJA^t = AJJA^{-1}J^{-1} = -J^{-1} = J$$

and

$$(A^tA)^t JA^tA = A^tAJA^tA = A^tJA = J$$

and therefore A^t and $R = A^tA$ are also symplectic. Also R is symmetric and $RJR = J$ or $JR = R^{-1}J$ and $RJ = JR^{-1}$. Moreover, R is positive definite (with respect to the euclidean inner product). So there exists a unique symmetric S such that $S^2 = R$, and $S \in \mathrm{Sp}(n, \mathbb{R})$, since

$$(-JS^{-1}J)^2 = -JS^{-2}J = -(JS^2J)^{-1} = -(JRJ)^{-1} = -(-R^{-1})^{-1} = R$$

and so $-JS^{-1}J = S$, by uniqueness. Now if $U = AS^{-1}$, then

$$\langle Ux, Uy\rangle = \langle S^{-1}x, RS^{-1}y\rangle = \langle S^{-1}x, Sy\rangle = \langle x, y\rangle$$

for every x, $y \in \mathbb{R}^{2n}$, since S is symmetric. Therefore, $U \in O(2n, \mathbb{R}) \cap \mathrm{Sp}(n, \mathbb{R}) = U(n)$ and we get the polar decomposition $A = US$. Since S is positive definite and symmetric, S^ρ is defined for all $\rho \geq 0$ and $S^{k/2^m} \in \mathrm{Sp}(n, \mathbb{R})$ for every $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. Since $\mathrm{Sp}(n, \mathbb{R})$ is a closed subset of $GL(2n, \mathbb{R})$ and $\{k/2^m : k \in \mathbb{N}$ and $m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$, we conclude that $S^\rho \in \mathrm{Sp}(n, \mathbb{R})$ for every $\rho \geq 0$. It follows now easily that $U(n)$ is a strong deformation retract of $\mathrm{Sp}(n, \mathbb{R})$. In particular, $\mathrm{Sp}(n, \mathbb{R})$ connected.

A n-dimensional subspace $W \leq \mathbb{R}^{2n}$ is Lagrangian if and only if $\omega|_W = 0$ or equivalently $J(W)$ is orthogonal to W with respect to the euclidean inner product $\langle , \rangle = \mathrm{Re}h$. Let $L(n)$ denote the set of all Lagrangian linear subspaces of $\mathbb{R}^{2n}$. Thus, the group $O(2n, \mathbb{R}) \cap GL(n, \mathbb{C})$ acts transitively on $L(n)$ and the isotropy group of the Lagrangian subspace $\mathbb{R}^n \times \{0\}$ is the subgroup of $U(n)$ consisting of real matrices, that is $O(n, \mathbb{R})$. Hence $L(n) = U(n)/O(n, \mathbb{R})$ and so it has the structure of a homogeneous smooth manifold of dimension

$$\dim U(n) - \dim O(n, \mathbb{R}) = n + \frac{2n(n-1)}{2} - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

For every $U \in U(n)$ and $A \in O(n,\mathbb{R})$ we have $\det(UA) = \det U \cdot \det A = \pm \det U$. So, we have a well defined smooth function $(\det)^2 : L(n) \to S^1$. Obviously, $(\det)^2$ is a submersion, and therefore a fibration, since $U(n)$ is compact. Let now U_1, $U_2 \in U(n)$ be such that $(\det U_1)^2 = (\det U_2)^2$. There exists $A \in O(n,\mathbb{R})$ such that $\det U_1 = \det(U_2 A)$ and therefore $U_1(U_2A)^{-1} \in SU(n)$ and

$$(U_1(U_2A)^{-1})U_2A \cdot O(n,\mathbb{R}) = U_1 \cdot O(n,\mathbb{R}).$$

Consequently, the group $SU(n)$ acts transitively on each fiber of $(\det)^2$ with isotropy group $SO(n,\mathbb{R})$. This shows that each fiber of $(\det)^2$ is diffeomorphic to the homogeneous space $SU(n)/SO(n,\mathbb{R})$, which is simply connected, since $SU(n)$ is simply connected and $SO(n,\mathbb{R})$ is connected, by the homotopy exact sequence of the fibration

$$SO(n,\mathbb{R}) \hookrightarrow SU(n) \to SU(n)/SO(n,\mathbb{R}).$$

From the homotopy exact sequence of the fibration

$$SU(n)/SO(n,\mathbb{R}) \hookrightarrow L(n) \xrightarrow{(\det)^2} S^1$$

we have the exact sequence

$$\{1\} = \pi_1(SU(n)/SO(n,\mathbb{R})) \to \pi_1(L(n)) \to \mathbb{Z} \to \pi_0(SU(n)/SO(n,\mathbb{R})) = \{1\}.$$

It follows that $(\det)^2$ induces an isomorphism $\pi_1(L(n)) \cong \mathbb{Z}$, and therefore also an isomorphism on singular cohomology $((\det)^2)^* : \mathbb{Z} \cong H^1(L(n);\mathbb{Z})$. The cohomology class $((\det)^2)^*(1)$ is called the Maslov class.

Remark 1.6. Note that $\det A = 1$ for every $A \in \mathrm{Sp}(n,\mathbb{R})$. Also if λ is an eigenvalue of A with multiplicity k, then

$$\det(A - \lambda I_{2n}) = \det(A^t - \lambda I_{2n}) = \det(J^{-1}(A^t - \lambda I_{2n})J) =$$

$$\det(A^{-1} - \lambda I_{2n}) = \det A^{-1} \cdot \det(I_{2n} - \lambda A) = \lambda^{2n} \cdot \det(A - \frac{1}{\lambda} I_{2n}).$$

So $\frac{1}{\lambda}$ is also an eigenvalue of A of multiplicity k.

2.2 Symplectic manifolds

A *symplectic manifold* is a pair (M,ω), where M is a smooth manifold and ω is a closed 2-form on M such that (T_pM, ω_p) is a symplectic vector space for every $p \in M$. Necessarily then there exists a positive integer n such that $\dim M = 2n$ and the n-fold wedge product $\omega \wedge ... \wedge \omega$ is a volume $2n$-form on M. So M is orientable and ω determines in this way an orientation.

A smooth map $f : (M,\omega) \to (M',\omega')$ between symplectic manifolds is called *symplectic* if $f^*\omega' = \omega$. If f is also a diffeomorphism, it is called *symplectomorphism*.

A submanifold N of M is called

(i) isotorpic if T_pN is an isotropic linear subspace of T_pM for every $p \in N$ and then $\dim N \leq n$,

(ii) coisotorpic if T_pN is a coisotropic linear subspace of T_pM for every $p \in N$ and then $\dim N \geq n$,

(iii) Lagrangian if T_pN is an Lagrangian linear subspace of T_pM for every $p \in N$ and then $\dim N = n$,

(iv) symplectic if T_pN is a symplectic linear subspace of T_pM for every $p \in N$ and then $\dim N$ is even.

Example 2.1. For every positive integer n, the space $\mathbb{R}^{2n}$ is a symplectic manifold, by considering on each tangent space $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ the canonical symplectic vector space structure. If dx^1, dx^2,..., dx^n, dy^1, dy^2,..., dy^n are the canonical basic differential 1-forms on $\mathbb{R}^{2n}$, then the canonical symplectic manifold structure is defined by the 2-form

$$\sum_{i=1}^{n} dx^i \wedge dy^i.$$

The basic example of a symplectic manifold is the cotangent bundle.

Example 2.2. Let M be a smooth n-manifold and $\pi : T^*M \to M$ be the cotangent bundle of M. There exists a canonical differential 1-form θ on T^*M, called the *Liouville 1-form*, defined by

$$\theta_a(v) = a(\pi_{*a}(v))$$

where $v \in T_a(T^*M)$ and $a \in T^*M$. We shall find a local formula for θ. Let $(U, q^1, q^2, ..., q^n)$ be a chart on M and $(\pi^{-1}(U), q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)$ be the corresponding chart on T^*M. Then,

$$\theta|_{\pi^{-1}(U)} = \sum_{i=1}^{n} \theta(\frac{\partial}{\partial q^i})dq^i + \sum_{i=1}^{n} \theta(\frac{\partial}{\partial p_i})dp_i.$$

If now $a = (q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)$, then $\pi_{*a}(\frac{\partial}{\partial p_i}) = 0$, and therefore $\theta(\frac{\partial}{\partial p_i}) = 0$. Moreover,

$$\theta(\frac{\partial}{\partial q^i}) = a(\pi_{*a}(\frac{\partial}{\partial q^i})) = p_i.$$

It follows that

$$\theta|_{\pi^{-1}(U)} = \sum_{i=1}^{n} p_i dq^i.$$

The canonical symplectic structure on T^*M is defined now by the closed 2-form $\omega = -d\theta$. Locally,

$$\omega|_{\pi^{-1}(U)} = \sum_{i=1}^{n} dq^i \wedge dp_i$$

and so ω is indeed non-degenerate. Note that $\omega|_{\pi^{-1}(U)}$ is the canonical symplectic form on $\mathbb{R}^{2n}$.

Another important example of a symplectic manifold is the complex projective space, which is an example of a Kähler manifold.

Example 2.3. For $n \geq 1$ let $\mathbb{C}P^n$ denote the complex projective space of complex dimension n and $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ be the quotient map. Recall that there is a canonical atlas $\{(V_j, \phi_j) : 0 \leq j \leq n\}$, where $V_j = \{[z_0, ..., z_n] \in \mathbb{C}P^n : z_j \neq 0\}$ and

$$\phi_j[z_0, ..., z_n] = (\frac{z_0}{z_j}, ..., \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, ..., \frac{z_n}{z_j}).$$

The quotient map π is a submersion. To see this note first that $\phi_0 \circ \pi : \pi^{-1}(V_0) \to \mathbb{C}^n$ is given by the formula

$$(\phi_0 \circ \pi)(z_0, ..., z_n) = (\frac{z_1}{z_0}, ..., \frac{z_n}{z_0}).$$

Let $z = (z_0, ..., z_n) \in \pi^{-1}(V_0)$ and $v = (v_0, ..., v_n) \in T_z\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ be non-zero. Then $v = \dot{\gamma}(0)$, where $\gamma(t) = z + tv$, and

$$(\phi_0 \circ \pi \circ \gamma)(t) = \left(\frac{z_1 + tv_1}{z_0 + tv_0}, ..., \frac{z_n + tv_n}{z_0 + tv_0}\right)$$

so that

$$(\phi_0 \circ \pi \circ \gamma)'(0) = \left(\frac{v_1}{z_0} - \frac{z_1 v_0}{z_0^2}, ..., \frac{v_n}{z_0} - \frac{z_n v_0}{z_0^2}\right).$$

This implies that $v \in \text{Ker } \pi_{*z}$ if and only if $[v_0, ..., v_n] = [z_0, ..., z_n]$. In other words $\text{Ker } \pi_{*z} = \{\lambda z : \lambda \in \mathbb{C}\}$. Obviously, for every $(\zeta_0, ..., \zeta_n) \in \mathbb{C}^n$ there exists $v = (v_0, ..., v_n) \in \mathbb{C}^{n+1}$ such that

$$\zeta_j = \frac{v_j}{z_0} - \frac{z_j v_0}{z_0^2}.$$

Since similar things hold for any other chart (V_j, ϕ_j) instead of (V_0, ϕ_0), this shows that π is a submersion.

Let h be the usual hermitian product on $\mathbb{C}^{n+1}$. If

$$W_z = \{\eta \in T_z\mathbb{C}^{n+1} : h(\eta, z) = 0\},$$

then $\pi_{*z}|_{W_z} : W_z \to T_{[z]}\mathbb{C}P^n$ is a linear isomorphism for every $z \in \mathbb{C}^{n+1} \setminus \{0\}$. Indeed, for every $v \in T_z\mathbb{C}^{n+1}$ there are unique $\lambda \in \mathbb{C}$ and $\eta \in W_z$ such that $v = \lambda z + \eta$. Obviously,

$$\lambda = \frac{h(v, z)}{h(z, z)} \cdot z, \qquad \eta = v - \frac{h(v, z)}{h(z, z)} \cdot z.$$

The restricted hermitian product on W_z can be transfered isomorphically by π_{*z} on $T_{[z]}\mathbb{C}P^n$. If now

$$g_{[z]}(v, w) = \text{Re } h((\pi_{*z}|_{W_z})^{-1}(v), (\pi_{*z}|_{W_z})^{-1}(w))$$

for v, $w \in T_{[z]}\mathbb{C}P^n$, then g is Riemannian metric on $\mathbb{C}P^n$ called the *Fubini-Study metric*. The scalar multiplication with the imaginary unit i defines a linear automorphism $J : W_z \to W_z$ such that $J^2 = -id$ and $h(Jv, Jw) = h(v, w)$ for every v, $w \in W_z$. Conjugating with $\pi_{*z}|_{W_z}$, we get a linear automorphism $J_{[z]}$ of $T_{[z]}\mathbb{C}P^n$ depending smoothly on $[z]$, which is a linear isometry, such that $J_{[z]}^2 = -id$. In other words, the Fubini-Study metric is a hermitian Riemannian metric.

If we set $\omega_{[z]}(v,w) = g_{[z]}(J_{[z]}v,w)$ for v, $w \in T_{[z]}\mathbb{C}P^n$, then

$$\omega_{[z]}(w,v) = g_{[z]}(J_{[z]}w,v) = g_{[z]}(v,J_{[z]}w) = -g_{[z]}(J_{[z]}v,w) = -\omega_{[z]}(v,w).$$

So we get a differential 2-form on $\mathbb{C}P^n$, which is obviously non-degenerate since $J_{[z]}$ is a linear isomorphism. To see that ω is symplectic, it remains to show that it is closed. This will be an application of the following general criterion.

Proposition 2.4. (Mumford's criterion) *Let M be a complex manifold and G be a group of diffeomorphisms of M which preserve the complex structure J of M and a complex hermitian metric h on M. If $J_z \in \rho(G_z)$ for every $z \in M$, where $\rho : G_z \to Aut_{\mathbb{C}}(T_zM)$ is the isotropic linear representation of the isotropy group G_z, then the 2-form $\omega(.,.) = \mathrm{Re}\ h(J.,.)$ is closed.*

Proof. Since every $g \in G$ leaves J and h invariant, it leaves ω invariant and therefore $d\omega$ also. For $g \in G_z$ and v, w, $u \in T_zM$ we have

$$(d\omega)_z(\rho_z(g)u, \rho_z(g)v, \rho_z(g)w) = (d\omega)_z(u,v,w)$$

because $\rho_z(g) = g_{*z}$. Since $J_z \in \rho_z(G_z)$, there exists $g \in G_z$ such that $J_z = \rho_z(g)$ and so

$$\begin{aligned}(d\omega)_z(u,v,w) = (d\omega)_z(J_zu, J_zv, J_zw) &= (d\omega)_z(J_z^2u, J_z^2v, J_z^2w)\\ &= -(d\omega)_z(u,v,w).\end{aligned}$$

Hence $d\omega = 0$. □

In the case of $\mathbb{C}P^n$ we apply Mumford's criterion for $G = SU(n+1)$, since $\mathbb{C}P^n$ is diffeomorphic to the homogeneous space $SU(n+1)/U(n)$. The group $SU(n+1)$ acts on $\mathbb{C}P^n$ in the usual way and $G_{[z]} = SU(n+1)_{[z]} \cong U(W_z)$. The isotropic linear representation here $\rho : SU(n+1)_{[z]} \to U(n)$ is precisely the above group isomorphism, since we deal with complex linear transformations. It follows that $J_{[z]} = iI_n \in U(W_z)$, and by Mumford's criterion ω is closed.

This concludes the description of the symplectic structure of complex projective spaces. One can give a similar description of the symplectic structure of general Kähler manifolds.

It should be noted that not every orientable, even-dimensional, smooth manifold carries a symplectic structure. If (M,ω) is a compact, symplectic manifold of dimension $2n$, then $[\omega]^n$ is a non-zero element of $H^{2n}_{DR}(M)$, the power taken with respect to the cup product. Thus, $[\omega]$ is a non-zero element of $H^2_{DR}(M)$. It follows that if M is an orientable, compact, smooth manifold such that $H^2_{DR}(M) = \{0\}$, then M admits no symplectic structure. For example, the n-sphere S^n cannot be symplectic for $n > 2$, as well as the 4-manifold $S^1 \times S^3$.

Having in mind the symplectic structure of the complex projective space we give the following.

Definition 2.5. An *almost symplectic structure* on a smooth manifold M of dimension $2n$ is non-degenerate, smooth 2-form on M. An *almost complex structure* on M is a smooth bundle endomorphism $J : TM \to TM$ such that $J^2 = -id$.

The following proposition leads to vector bundle obstructions for a compact manifold to be symplectic.

Proposition 2.6. *A smooth manifold M of dimension $2n$ has an almost complex structure if and only if it has an almost symplectic structure.*

Proof. Let J be an almost complex structure on M. Let g_0 be any Riemannian metric on M and g be the Riemannian metric defined by

$$g(v, w) = g_0(v, w) + g_0(Jv, Jw)$$

for v, $w \in T_pM$, $p \in M$. Then,

$$g(Jv, Jw) = g_0(Jv, Jw) + g_0(J^2v, J^2w) = g_0(Jv, Jw) + g_0(-v, -w) = g(v, w).$$

The smooth 2-form ω defined by

$$\omega(v, w) = g(Jv, w)$$

is non-degenerate, because $\omega(v, Jv) > 0$ for $v \neq 0$.

For the converse, suppose that ω is an almost symplectic structure on M and let again g be any Riemannian metric on M. There exists a smooth bundle endomorphism $A : TM \to TM$ (depending on g) such that $\omega(v, w) = g(Av, w)$ for all v, $w \in T_pM$, $p \in M$. Since ω is non-degenerate and skew-symetric, A is an automorphism and skew-symetric (with respect to g). Therefore, $-A^2$ is positive definite and symmetric (with respect to g). So, it has a unique square root, which means that there is a unique smooth bundle automorphism $B : TM \to TM$ such that $B^2 = -A^2$. Moreover, B commutes with A. Then, $J = AB^{-1}$ is an almost complex structure on M. $\square$

If now M is a compact symplectic manifold of dimension $2n$, it has an almost complex structure. Hence its tangent bundle is the realification of a unique complex vector bundle. In other words, the tangent bundle of M can be considered as a complex vector bundle whose fiber has complex dimension n, to which correspond Chern classes $c_k \in H^{2k}(M; \mathbb{Z})$, $1 \leq k \leq n$. The Chern classes are related to the Pontryagin classes of the tangent bundle of M through polynomial (quadratic) equations, which can serve as obstructions to the existence of a symplectic structure on M, since not every compact, orientable, smooth $2n$-manifold has cohomology classes satisfying these equations. For instance, using these equations and Hirzebruch's Signature Theorem, one can show that the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ cannot be a symplectic manifold.

2.3 Local description of symplectic manifolds

Even though we have defined the symplectic structure in analogy to the Riemannian structure, their local behaviour differs drastically. In this section we shall show that in the neighbourhood of any point on a symplectic $2n$-manifold (M, ω) there are suitable local coordinates $(q^1, ..., q^n, p_1, ..., p_n)$ such that

$$\omega|_{\text{locally}} = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

This shows that in symplectic geometry there no local invariants, in contrast to Riemannian geometry, where there are highly non-trivial local invariants. In other words, the study of symplectic manifolds is of global nature and one expects to use mainly topological methods.

The method of proof of the local isomorphy of all symplectic manifolds, we shall present, is based on Moser's trick.

Lemma 3.1. (Moser) *Let M and N be two smooth manifolds and $F : M \times \mathbb{R} \to N$ be a smooth map. For every $t \in \mathbb{R}$ let $X_t : M \to TN$ be the smooth vector field along $F_t = F(., t)$ defined by*

$$X_t(p) = \left.\frac{\partial}{\partial s}\right|_{s=t} F(p, s) \in T_{F_t(p)}N.$$

If $(\omega_t)_{t \in \mathbb{R}}$ is a smooth family of k-forms on N, then

$$\frac{d}{dt}(F_t^* \omega_t) = F_t^*(\frac{d\omega_t}{dt} + i_{X_t} d\omega_t) + d(F_t^* i_{X_t} \omega_t).$$

If moreover F_t is a diffeomorphism for every $t \in \mathbb{R}$, then

$$\frac{d}{dt}(F_t^* \omega_t) = F_t^*(\frac{d\omega_t}{dt} + i_{X_t} d\omega_t + d i_{X_t} \omega_t).$$

Note that if F_t is not a diffeomorphism then X_t is not in general a vector field on N. The meaning of the symbol $F_t^* i_{X_t} \omega_t$ will be clear in the proof.

Proof. (a) First we shall prove the formula in the special case $M = N = P \times \mathbb{R}$ and $F_t = \psi_t$, where $\psi_t(x, s) = (x, s + t)$. Then

$$\omega_t = ds \wedge a(x, s, t) dx^k + b(x, s, t) dx^{k+1},$$

where

$$a(x, s, t) dx^k = \sum_{i_1 < i_2 < ... < i_k} a_{i_1 i_2 ... i_k}(x, s, t) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

and similarly for $b(x, s, t) dx^{k+1}$. So $\psi_t^* \omega_t = ds \wedge a(x, s+t, t) dx^k + b(x, s+t, t) dx^{k+1}$ and

$$\frac{d}{dt}(\psi_t^* \omega_t) = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t) dx^k + \frac{\partial b}{\partial s}(x, s+t, t) dx^{k+1}$$

$$+ ds \wedge \frac{\partial a}{\partial t}(x, s+t, t) dx^k + \frac{\partial b}{\partial t}(x, s+t, t) dx^{k+1},$$

Obviously,

$$\psi_t^*(\frac{d\omega_t}{dt}) = ds \wedge \frac{\partial a}{\partial t}(x, s+t, t) dx^k + \frac{\partial b}{\partial t}(x, s+t, t) dx^{k+1}. \tag{1}$$

On the other hand $X_t = \dfrac{\partial}{\partial s}$. So $i_{X_t}\omega_t = a(x,s,t)dx^k$ and

$$d(i_{X_t}\omega_t) = \sum_{i_1<i_2<\ldots<i_k} da_{i_1 i_2 \ldots i_k}(x,s,t) \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} =$$

$$\sum_{i_1<i_2<\ldots<i_k} \left(\frac{\partial a_{i_1 i_2 \ldots i_k}}{\partial s}(x,s,t) ds \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} + \right.$$

$$\left. \sum_{j \notin \{i_1<i_2<\ldots<i_k\}} \frac{\partial a_{i_1 i_2 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \right).$$

We shall write for brevity

$$d(i_{X_t}\omega_t) = \frac{\partial a}{\partial s}(x,s,t) ds \wedge dx^k + d_x a(x,s,t) dx^{k+1}.$$

So

$$\psi_t^*(d(i_{X_t}\omega_t)) = \frac{\partial a}{\partial s}(x,s+t,t) ds \wedge dx^k + d_x a(x,s+t,t) dx^{k+1}. \tag{2}$$

Using the symbol d_x in the same way, we have

$$d\omega_t = -ds \wedge d_x a(x,s,t) dx^k + \frac{\partial b}{\partial s}(x,s,t) ds \wedge dx^{k+1} + d_x b(x,s,t) dx^{k+2},$$

and thus

$$\psi_t^*(i_{X_t} d\omega_t) = -d_x a(x,s+t,t) dx^{k+1} + \frac{\partial b}{\partial s}(x,s+t,t) dx^{k+1}. \tag{3}$$

Summing up now (1), (2) and (3) we get

$$\psi_t^*(\frac{d\omega_t}{dt}) + \psi_t^* d(i_{X_t}\omega_t) + \psi_t^*(i_{X_t} d\omega_t) = \frac{d}{dt}\psi_t^*\omega_t.$$

(b) The general case follows from part (a) using the decomposition $F_t = F \circ \psi_t \circ j$, where $j : M \to M \times \mathbb{R}$ is the inclusion $j(p) = (0,p)$ and ψ_t is the same as in part (a). Now we have

$$X_t(p) = \left.\frac{\partial}{\partial s}\right|_{s=t} F(p,s) = F_{*(p,t)}\left(\frac{\partial}{\partial s}\right)_{(p,t)}.$$

If each F_t is a diffeomorphism, then X_t is a vector field on N and $i_{X_t}\omega_t$ is defined. If not, the term $F_t^*(i_{X_t}\omega_t)$ has the following meaning. By definition,

$$F_t^*(i_{X_t}\omega_t)_p(v_1, \ldots, v_{k-1}) = (\omega_t)_{F_t(p)}(X_t(p), (F_t)_{*p}(v_1), \ldots, (F_t)_{*p}(v_{k-1})) =$$

$$(\omega_t)_{F(p,t)}(F_{*(p,t)}\left(\frac{\partial}{\partial s}\right)_{(p,t)}, F_{*(p,t)}(v_1,0), \ldots, F_{*(p,t)}(v_{k-1},0)) =$$

$$(F^*\omega_t)_{(p,t)}(\left(\frac{\partial}{\partial s}\right)_{(p,t)}, (v_1,0), \ldots, (v_{k-1},0)) = (i_{\partial/\partial s}F^*\omega_t)_{(p,t)}((v_1,0), \ldots, (v_{k-1},0)) =$$

$$j^*\psi_t^*(i_{\partial/\partial s}F^*\omega_t)_p(v_1,...,v_{k-1})$$

for v_1,...,$v_{k-1} \in T_pM$. Therefore, $F_t^*(i_{X_t}\omega_t) = j^*\psi_t^*(i_{\partial/\partial s}F^*\omega_t)$ and similarly $F_t^*(i_{X_t}d\omega_t) = j^*\psi_t^*(i_{\partial/\partial s}d(F^*\omega_t))$. Since j^* does not depend on t, we have

$$\frac{d}{dt}(F_t^*\omega_t) = j^*\frac{d}{dt}(\psi_t^*F^*\omega_t)$$

and applying part (a) to $F^*\omega_t$ we get

$$\frac{d}{dt}(F_t^*\omega_t) = j^*\psi_t^*(\frac{d(F^*\omega_t)}{dt}) + j^*\psi_t^*(i_{\partial/\partial s}d(F^*\omega_t)) + j^*d(\psi_t^*i_{\partial/\partial s}(F^*\omega_t)) =$$

$$j^*\psi_t^*F^*(\frac{d\omega_t}{dt}) + F_t^*(i_{X_t}d\omega_t) + d(F_t^*(i_{X_t}\omega_t)) =$$

$$F_t^*(\frac{d\omega_t}{dt}) + F_t^*(i_{X_t}d\omega_t) + d(F_t^*(i_{X_t}\omega_t)). \quad \square$$

Corollary 3.2. *Let X be a smooth vector field on a smooth manifold M. If ω is a differential form on M, then $L_X\omega = i_Xd\omega + di_X\omega$.*

Proof. If X is complete and $(\phi_t)_{t\in\mathbb{R}}$ is its flow, we apply Lemma 3.1 for $F_t = \phi_t$, $M = N$ and $\omega_t = \omega$ and we have

$$L_X\omega = \frac{d}{dt}\bigg|_{t=0}\phi_t^*\omega = i_Xd\omega + di_X\omega.$$

If X is not complete, then M has an open covering $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists some $\epsilon > 0$ and a local flow map $\phi : (-\epsilon,\epsilon)\times U \to M$ of X. Again we apply Lemma 3.1 for $F_t = \phi_t$ on U this time to get the desired formula on every $U \in \mathcal{U}$, hence on M. $\square$

We are now in a position to prove the main theorem of this section.

Theorem 3.3. (Darboux) *Let ω_0 and ω_1 be two symplectic 2-forms on a smooth $2n$-manifold M and $p \in M$. If $\omega_0(p) = \omega_1(p)$, there exists an open neighbourhood U of p in M and a diffeomorphism $F : U \to F(U) \subset M$, where $F(U)$ is an open neighbourhood of p, such that $F(p) = p$ and $F^*\omega_1 = \omega_0$.*

Proof. Let $\omega_t = (1-t)\omega_0 + t\omega_1$, $0 \le t \le 1$. Since $\omega_t(p) = \omega_0(p) = \omega_1(p)$, there exists an open neighbourhood U_1 of p diffeomorphic to $\mathbb{R}^{2n}$ such that $\omega_t|_{U_1}$ is symplectic for every $0 \le t \le 1$. By the lemma of Poincaré, there exists a 1-form a on U_1 such that $\omega_0 - \omega_1 = da$ on U_1 and $a(p) = 0$. For every $0 \le t \le 1$ there exists a smooth vector field Y_t on U_1 such that $i_{Y_t}\omega_t = a$. Obviously, $Y_t(p) = 0$ and the above hold for every $-\epsilon < t < 1+\epsilon$, for some $\epsilon > 0$. Now $\bar{Y} = (\frac{\partial}{\partial s}, Y_s(p))$ is a smooth vector field on $(-\epsilon, 1+\epsilon)\times U_1$. If ϕ_t is the flow of $\bar{Y}$, then $\phi_t(s,x) = (s+t, f_t(s,x))$, for some smooth $f_t : U_1 \to M$. Therefore, $\phi_t(0,x) = (t, F_t(x))$, where $F_t : U_1 \to F_t(U_1)$ is a diffeomorphism. Since $\phi_t(0,p) = (0,p)$, that is $F_t(p) = p$, there exists an open neighbourhood U of p such that F_t is defined on U and $F_t(U) \subset U_1$ for every $0 \leq t \leq 1$. Obviously, $Y_t = \dfrac{\partial F_t}{\partial t}$ and so from Lemma 3.1 we have

$$\frac{d}{dt}(F_t^* \omega_t) = F_t^*(\frac{d\omega_t}{dt} + i_{Y_t} d\omega_t + d i_{Y_t}\omega_t) = F_t^*(\omega_1 - \omega_0 + 0 + da) = 0.$$

Hence $F_t^*\omega_t = F_0^*\omega_0 = \omega_0$ for every $0 \leq t \leq 1$, since $F_0 = id$. □

Corollary 3.4. *Let (M, ω) be a symplectic $2n$-manifold and $p \in M$. There exists an open neighbourhood U of p and a diffeomorphism $F : U \to F(U) \subset \mathbb{R}^{2n}$ such that*

$$\omega|_U = F^*\big(\sum_{i=1}^{n} dx^i \wedge dy^i\big).$$

Proof. Let (W, ψ) be a chart of M with $p \in W$, $\psi(W) = \mathbb{R}^{2n}$ and $\psi(p) = 0$. Then the 2-form $\omega_1 = (\psi^{-1})^*\omega$ on $\mathbb{R}^{2n}$ is symplectic. Composing with a linear transformation if necessary, we may assume that $\omega_1(0) = \omega_0(0)$, where ω_0 is the standard symplectic 2-form on $\mathbb{R}^{2n}$. By Darboux's theorem, there exists an open neighbourhood V of 0 in $\mathbb{R}^{2n}$ and a diffeomorphism $\phi : V \to \phi(V)$ with $\phi(0) = 0$ and $\phi^*\omega_1 = \omega_0$. It suffices to set now $F = (\psi^{-1} \circ \phi)^{-1}$. □

At this point we cannot resist the temptation to use Moser's trick in order to prove the following result, also due to J. Moser.

Theorem 3.5. (Moser) *Let M be a connected, compact, oriented, smooth n-manifold and ω_0, ω_1 be two representatives of the orientation. If*

$$\int_M \omega_0 = \int_M \omega_1,$$

there exists a diffeomorphism $f : M \to M$ such that $f^*\omega_1 = \omega_0$.

Proof. For every $0 \leq t \leq 1$ the n-form $\omega_t = (1-t)\omega_0 + t\omega_0$ is a representative of the orientation, that is a positive volume element of M. Since

$$\int_M (\omega_0 - \omega_1) = 0,$$

there exists a $(n-1)$-form a on M such that $\omega_0 - \omega_1 = da$. There exists a unique smooth vector field X_t on M such that $i_{X_t}\omega_t = a$. As in the proof of Darboux's theorem, there exists a smooth isotopy $F : M \times [0,1] \to M$ with $F_0 = id$ and

$$X_t = \frac{\partial F_t}{\partial t},$$

because M is compact. Again from Lemma 3.1 we have

$$\frac{d}{dt}(F_t^*\omega_t) = F_t^*(\omega_1 - \omega_0 - 0 + da) = 0.$$

Hence $F_t^*\omega_t = \omega_0$ for every $0 \leq t \leq 1$. □

2.4 Hamiltonian vector fields and Poisson bracket

Let (M, ω) be a symplectic $2n$-manifold. A smooth vector field X on M is called *Hamiltonian* if there exists a smooth function $H : M \to \mathbb{R}$ such that $i_X\omega = dH$. In other words,

$$\omega_p(X_p, v_p) = v_p(H)$$

for every $v_p \in T_pM$ and $p \in M$. We usually write $X = X_H$ and obviously $X_H = \tilde{\omega}^{-1}(dH)$.

If $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with the canonical symplectic 2-form

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i,$$

we have $\tilde{\omega}(\frac{\partial}{\partial q^i}) = dp_i$ and $\tilde{\omega}(\frac{\partial}{\partial p^i}) = -dq^i$. Thus,

$$X_H = \tilde{\omega}^{-1}\Big(\sum_{i=1}^{n} \frac{\partial H}{\partial q^i} dq^i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i\Big) = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial p_i} \cdot \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \cdot \frac{\partial}{\partial p_i}\Big).$$

So the integral curves of X_H are the solutions of Hamilton's differential equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad 1 \leq i \leq n.$$

According to Darboux's theorem, this is true locally, with respect to suitable local coordinates, on every symplectic $2n$-manifold.

A smooth vector field X on M is called *symplectic* or *locally Hamiltonian* if $L_X\omega = d(i_X\omega) = 0$. In this case the lemma of Poincaré implies that every point $p \in M$ has an open neighbourhood V diffeomorphic to $\mathbb{R}^{2n}$ for which there exists a smooth function $H_V : V \to \mathbb{R}$ such that $i_X\omega|_V = dH_V$. From Lemma 3.1, we have

$$\frac{d}{dt}\phi_t^*\omega = \phi_t^*(d(i_X\omega)),$$

and $\phi_0^*\omega = \omega$, where ϕ_t is the flow of X. Thus X is locally Hamiltonian if and only if its flow consists of symplectomorphisms.

A locally Hamiltonian vector field may not be Hamiltonian. As a simple example, let $M = S^1 \times S^1$ equiped with the volume element ω such that $\pi^*\omega = dx \wedge dy$, where $\pi : \mathbb{R}^2 \to M$ is the universal covering projection. The smooth vector field

$$X = \pi_*(\frac{\partial}{\partial x})$$

is locally Hamiltonian, since locally $\pi^*(i_X\omega) = dy$. But if $j : S^1 \to M$ is the embedding $j(z) = (1, z)$, then $j^*(i_X\omega)$ is the natural generator of $H^1_{DR}(S^1) \cong \mathbb{R}$ and thus it is not exact. Therefore $i_X\omega$ is not exact.

Two elementary properties of Hamiltonian vector fields are the following.

Proposition 4.1. *The smooth function $H : M \to \mathbb{R}$ is a first integral of the Hamiltonian vector field X_H.*

Proof. Indeed $X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$. □

Proposition 4.2. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is symplectic if and only if $f_*(X_{H\circ f}) = X_H$ for every open set $U \subset M_2$ and smooth function $H : U \to \mathbb{R}$.*

Proof. The condition $X_H(f(p)) = f_{*p}(X_{H\circ f}(p))$ for every $p \in f^{-1}(U)$ is equivalent to

$$\omega_2(f(p))(X_H(f(p)), f_{*p}(v)) = \omega_2(f(p))(f_{*p}(X_{H\circ f}(p)), f_{*p}(v))$$

for every $v \in T_pM$, since ω_2 is non-degenerate and f is a diffeomorphism. Equivalently,

$$dH(f(p))(f_{*p}(v)) = (f^*\omega_2)(p)(X_{H\circ f}(p), v)$$

or

$$i_{X_{H\circ f}}\omega_1 = d(H \circ f) = f^*(dH) = i_{X_{H\circ f}}(f^*\omega_2)$$

on $f^{-1}(U)$. This is true, if f is symplectic. Conversely, if this holds, then for every $p \in M_1$ and u, $v \in T_pM_1$ there exists an open neighbourhood U of $f(p)$ in M_2 and a smooth function $H : U \to \mathbb{R}$ such that $u = X_{H\circ f}(p)$. So, $\omega_1(p)(u, v) = (f^*\omega_2)(p)(u, v)$ for every u, $v \in T_pM_1$. This means that $f^*\omega_2 = \omega_1$. □

If (M, ω) is a symplectic manifold and F, $G \in C^\infty(M)$, then the smooth function

$$\{F, G\} = i_{X_G} i_{X_F}\omega \in C^\infty(M)$$

is called the *Poisson bracket* of F and G. From Proposition 4.2 we obtain the following.

Corollary 4.3. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is symplectic if and only if*

$$f^*\{F, G\} = \{f^*(F), f^*(G)\}$$

for every open set $U \subset M_2$ and F, $G \in C^\infty(U)$.

Proof. Suppose first that f is a symplectomorphism and F, $G \in C^\infty(U)$, where $U \subset M_2$ is an open set. Then

$$\{F, G\}(f(p)) = \omega_2(f(p))(X_F(f(p)), X_G(f(p)))$$
$$= \omega_2(f(p))(f_{*p}(X_{F\circ f}(p)), f_{*p}(X_{G\circ f}(p))) = (f^*\omega_2)(p)(X_{F\circ f}(p), X_{G\circ f}(p))$$
$$= \omega_1(p)(X_{F\circ f}(p), X_{G\circ f}(p)) = \{F \circ f, G \circ f\}(p).$$

Conversely, if $\{F, G\}(f(p)) = \{F \circ f, G \circ f\}(p)$ for every $p \in f^{-1}(U)$ and every open set $U \subset M_2$ and F, $G \in C^\infty(U)$, then

$$(f^*\omega_2)(p)(X_{F\circ f}(p), X_{G\circ f}(p)) = \omega_1(p)(X_{F\circ f}(p), X_{G\circ f}(p)).$$

But for every $p \in M_1$ and u, $v \in T_pM_1$ there exists an open set $V \subset M_1$ with $p \in V$ and F, $G \in C^\infty(f(V))$ such that $X_{F\circ f}(p) = u$ and $X_{G\circ f}(p) = v$. So $(f^*\omega_2)(p)(u,v) = \omega_1(p)(u,v)$. This means $f^*\omega_2 = \omega_1$. □

Corollary 4.4. *Let X be a complete Hamiltonian vector field with flow $(\phi_t)_{t\in\mathbb{R}}$ on a symplectic manifold M. Then $\phi_t^*\{F,G\} = \{\phi_t^*(F), \phi_t^*(G)\}$ for every F, $G \in C^\infty(M)$. If X is not complete, the same is true on suitable open sets.*

Corollary 4.5. *Let X be a complete Hamiltonian vector field with flow $(\phi_t)_{t\in\mathbb{R}}$ on a symplectic manifold (M,ω). Then*

$$X\{F,G\} = \{X(F),G\} + \{F,X(G)\}$$

for every F, $G \in C^\infty(M)$. If X is not complete, the same is true on suitable open sets.

Proof. From Corollary 4.4 we have

$$X\{F,G\} = \frac{d}{dt}\bigg|_{t=0} \phi_t^*\{F,G\} = \frac{d}{dt}\bigg|_{t=0} \{\phi_t^*(F), \phi_t^*(G)\} = \frac{d}{dt}\bigg|_{t=0} \omega(X_{\phi_t^*(F)}, X_{\phi_t^*(G)})$$

$$= \omega(\frac{d}{dt}\bigg|_{t=0} X_{\phi_t^*(F)}, X_G) + \omega(X_F, \frac{d}{dt}\bigg|_{t=0} X_{\phi_t^*(G)}).$$

But

$$\tilde{\omega}(\frac{d}{dt}\bigg|_{t=0} X_{\phi_t^*(F)}) = \frac{d}{dt}\bigg|_{t=0} \tilde{\omega}(X_{\phi_t^*(F)}) = \frac{d}{dt}\bigg|_{t=0} d\phi_t^*(F)$$

$$= d(\frac{d}{dt}\bigg|_{t=0} \phi_t^*(F)) = dX(F) = \tilde{\omega}(X_{X(F)}),$$

which means that

$$\frac{d}{dt}\bigg|_{t=0} X_{\phi_t^*(F)} = X_{X(F)}.$$

Consequently,

$$X\{F,G\} = \omega(X_{X(F)}, X_G) + \omega(X_F, X_{X(G)}) = \{X(F),G\} + \{F,X(G)\}. \qquad \square$$

It is obvious that for a symplectic manifold (M,ω) the Poisson bracket

$$\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

is bilinear and skew-symmetric. Form Corollary 4.5 follows now that it satisfies the Jacobi identity. Indeed, if F, G, $H \in C^\infty(M)$ then

$$\{F,G\} = (i_{X_F}\omega)(X_G) = dF(X_G) = X_G(F)$$

and thus $\{\{F,G\},H\} = X_H(\{F,G\})$. Consequently,

$$\{\{F,G\},H\} = \{X_H(F),G\} + \{F,X_H(G)\} = \{\{F,H\},G\} + \{F,\{G,H\}\}.$$

This is the Jacobi identity and so the $(C^\infty(M), \{,\})$ is a Lie algebra.

There is a Leibniz formula for the product of two smooth functions with respect to the Poisson bracket, because if F, G, $H \in C^\infty(M)$, then

$$\{F \cdot G, H\} = X_H(F \cdot G) = F \cdot X_H(G) + G \cdot X_H(F) = F \cdot \{G, H\} + G \cdot \{F, H\}.$$

Proposition 4.6. *Let X_H be a Hamiltonian vector field with flow ϕ_t on a symplectic manifold M. Then*

$$\frac{d}{dt}(F \circ \phi_t) = \{F \circ \phi_t, H\} = \{F, H\} \circ \phi_t$$

for every $F \in C^\infty(M)$.

Proof. By the chain rule, for every $p \in M$ we have

$$\frac{d}{dt}(F \circ \phi_t)(p) = (dF)(\phi_t(p))X_H(\phi_t(p)) = \{F, H\}(\phi_t(p))$$

$$= \{F \circ \phi_t, H \circ \phi_t\}(p) = \{F \circ \phi_t, H\}(p),$$

since H is a first integral of X_H. □

Corolary 4.7. *A smooth function $F : M \to \mathbb{R}$ on a symplectic manifold M is a first integral of a Hamiltonian vector field X_H on M if and only if $\{F, H\} = 0$.* □

Let $\mathfrak{sp}(M, \omega)$ and $\mathfrak{h}(M, \omega)$ denote the linear spaces of the symplectic and hamiltonian vector fields, respectively, of the symplectic manifold (M, ω). We shall conclude this section with a few remarks about these spaces.

Proposition 4.8. *If X, $Y \in \mathfrak{sp}(M, \omega)$, then $[X, Y] = -X_{\omega(X,Y)}$. In particular, $[X_F, X_G] = -X_{\{F,G\}}$ for every F, $G \in C^\infty(M)$.*

Proof. Indeed, $i_{[X,Y]} = [L_X, i_Y]$ and therefore

$$i_{[X,Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = d(i_X i_Y \omega) + i_X(d(i_Y\omega)) - 0 =$$

$$d(\omega(Y, X)) + 0 = -i_{X_{\omega(X,Y)}}\omega.$$

Since ω is non-degenerate the result follows. □

Proposition 4.8 implies that $\mathfrak{sp}(M, \omega)$ and $\mathfrak{h}(M, \omega)$ are Lie subalgebras of the Lie algebra of smooth vector fields of M. Moreover, $\mathfrak{h}(M, \omega)$ is an ideal in $\mathfrak{sp}(M, \omega)$ since $[\mathfrak{sp}(M, \omega), \mathfrak{sp}(M, \omega)] \subset \mathfrak{h}(M, \omega)$. If M is connected, we have two obvious short exact sequences of Lie algebra homomorphisms

$$0 \to \mathbb{R} \to C^\infty(M) \xrightarrow{r} \mathfrak{h}(M, \omega) \to 0,$$

where $r(F) = -X_F$ for every $F \in C^\infty(M)$, and

$$0 \to \mathfrak{h}(M, \omega) \to \mathfrak{sp}(M, \omega) \to H^1_{DR}(M) \to 0,$$

which do not split in general. The first makes $(C^\infty(M), \{,\})$ a central extension of $\mathfrak{h}(M,\omega)$. In the second, we consider in $H^1_{DR}(M)$ the trivial Lie bracket.

Proposition 4.9. *Let (M,ω) be a compact, connected, symplectic $2n$-manifold and $\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$ (n times).*

(a) If X, $Y \in \mathcal{X}(M)$, then $\omega(X,Y)\omega^n = -n \cdot i_X\omega \wedge i_Y\omega \wedge \omega^{n-1}$.

(b) If F, $G \in C^\infty(M)$, then

$$\int_M \{F,G\}\omega^n = 0.$$

(c) The set $C_0^\infty(M,\omega) = \{F \in C^\infty(M) : \int_M F\omega^n = 0\}$ is a Lie subalgebra of $(C^\infty(M), \{,\})$ and $C^\infty(M) = \mathbb{R} \oplus C_0^\infty(M,\omega)$.

(d) The short exact sequence of Lie algebra homomorphisms

$$0 \to \mathbb{R} \to C^\infty(M) \xrightarrow{r} \mathfrak{h}(M,\omega) \to 0$$

splits.

Proof. (a) Since

$$0 = i_X(i_Y\omega \wedge \omega^n) = \omega(X,Y)\omega^n - i_Y\omega \wedge i_X\omega^n$$

and $i_X\omega^n = n \cdot i_X\omega \wedge \omega^{n-1}$, we conclude that

$$\omega(X,Y)\omega^n = n \cdot i_Y\omega \wedge i_X\omega \wedge \omega^{n-1}.$$

(b) Using (a) and the fact that ω^{n-1} is closed, we have

$$\{F,G\}\omega^n = \omega(X_F,X_G)\omega^n = -n \cdot i_{X_F}\omega \wedge i_{X_G}\omega \wedge \omega^{n-1} =$$

$$-n \cdot dF \wedge dG \wedge \omega^{n-1} = -n \cdot d(FdG \wedge \omega^{n-1}).$$

The conclusion follows now from Stokes formula.

(c) From (b) follows immediately that $C_0^\infty(M,\omega)$ is a Lie subalgebra of $(C^\infty(M), \{,\})$. Moreover, every $F \in C^\infty(M)$ can be written as

$$F = \frac{1}{\mathrm{vol}(M)} \int_M F\omega^n + \left(F - \frac{1}{\mathrm{vol}(M)} \int_M F\omega^n\right).$$

(d) If we define $j : \mathfrak{h}(M,\omega) \to C^\infty(M)$ by

$$j(X_F) = -F + \frac{1}{\mathrm{vol}(M)} \int_M F\omega^n,$$

then $\{j(X_F), j(X_G)\} = \{F,G\} = j(-X_{\{F,G\}}) = j([X_F,X_G])$, by Proposition 4.8, and therefore j is a Lie algebra homomorphism. Obviously, $r \circ j = id$. □

2.5 Coadjoint orbits

Let G be a Lie group with Lie algera $\mathfrak{g}$ and identity element e. The action of G on itself by conjugation, i.e. $\psi_g(h) = ghg^{-1}$, $g \in G$, fixes e and induces the *adjoint linear representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ defined by

$$\mathrm{Ad}_g(X) = (\psi_g)_{*e}(X) = \frac{d}{dt}\bigg|_{t=0} g(\exp tX)g^{-1}.$$

Example 5.1. The Lie group $SO(3, \mathbb{R})$ is compact, connected and its Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is isomorphic to the Lie algebra of skew-symmetric linear maps of $\mathbb{R}^3$ with respect to the Lie bracket $[A, B] = AB - BA$, A, $B \in \mathbb{R}^{3\times3}$.

On the other hand, the map $\widehat{\ }\ : \mathbb{R}^3 \to \mathfrak{so}(3, \mathbb{R})$ defined by

$$\hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

where $v = (v_1, v_2, v_3)$, is a linear isomorphism and $\hat{v} \cdot w = v \times w$, for every v, $w \in \mathbb{R}^3$. This actually characterizes $\widehat{\ }$. So we have

$$(\hat{u}\hat{v} - \hat{v}\hat{u})w = \hat{u}(v \times w) - \hat{v}(u \times w) = u \times (v \times w) - v \times (u \times w) = (u \times v) \times w = (\widehat{u \times v})w.$$

Thus, $\widehat{\ }$ is a Lie algebra isomorphism of the Lie algebra $(\mathbb{R}^3, \times)$ onto $\mathfrak{so}(3, \mathbb{R})$. Using this isomorphism we can describe the exponential map of $SO(3, \mathbb{R})$.

Let $w \in \mathbb{R}^3$, $w \neq 0$, and $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ such that $e_1 = w/\|w\|$. The matrix of $\hat{w}$ with respect to this basis is

$$\hat{w} = \|w\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $t \in \mathbb{R}$ let $\gamma(t)$ be the rotation around the axis determined by w through the angle $t\|w\|$, that is

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t\|w\| & -\sin t\|w\| \\ 0 & \sin t\|w\| & \cos t\|w\| \end{pmatrix}.$$

Then,

$$\dot{\gamma}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\|w\| \sin t\|w\| & -\|w\| \cos t\|w\| \\ 0 & \|w\| \cos t\|w\| & -\|w\| \sin t\|w\| \end{pmatrix} = \gamma(t)\hat{w} = (L_{\gamma(t)})_{*I_3}(\hat{w}) = X_{\hat{w}}(\gamma(t)),$$

where $L_{\gamma(t)}$ denotes the left translation on $SO(3, \mathbb{R})$ by $\gamma(t)$ and $X_{\hat{w}}$ the left invariant vector field on $SO(3, \mathbb{R})$ corresponding to $\hat{w}$. In other words, γ is an integral curve of $X_{\hat{w}}$ with $\gamma(0) = I_3$. It follows that $\exp(t\hat{w}) = \gamma(t)$ for every $t \in \mathbb{R}$.

For every $A \in SO(3, \mathbb{R})$ and $v \in \mathbb{R}^3$ we have now

$$\mathrm{Ad}_A(\hat{v}) = \frac{d}{dt}\bigg|_{t=0} A(\exp(t\hat{v}))A^{-1} = \frac{d}{dt}\bigg|_{t=0} A\gamma(t)A^{-1} = A\gamma(0)\hat{v}A^{-1} = A\hat{v}A^{-1}.$$

Thus,

$$\mathrm{Ad}_A(\hat{v})w = A\hat{v}(A^{-1}w) = A(v \times A^{-1}w) = Av \times w$$

for every $w \in \mathbb{R}^3$, since $\det A = 1$. Hence $\mathrm{Ad}_A(\hat{v}) = \widehat{Av}$, and identifying $\mathbb{R}^3$ with $\mathfrak{so}(3,\mathbb{R})$ via $\widehat{\ }$ we conclude that $\mathrm{Ad}_A = A$.

Let now $\mathrm{ad} = (\mathrm{Ad})_{*e} : \mathfrak{g} \to T_e\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$, that is

$$\mathrm{ad}_X = (\mathrm{Ad})_{*e}(X) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}$$

for every $X \in \mathfrak{g}$. If we denote by X_L the left invariant vector field corresponding to X and $(\phi_t)_{t\in\mathbb{R}}$ its flow, then for every $Y \in \mathfrak{g} \cong T_0\mathfrak{g}$ we have

$$\mathrm{ad}_X(Y) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y) = \left.\frac{d}{dt}\right|_{t=0} (\psi_{\exp(tX)})_{*e}(Y) =$$

$$\left.\frac{d}{dt}\right|_{t=0} (R_{\exp(-tX)} \circ L_{\exp(tX)})_{*e}(Y) = \left.\frac{d}{dt}\right|_{t=0} (R_{\exp(-tX)})_{*\exp(tX)} \circ (L_{\exp(tX)})_{*e}(Y) =$$

$$\left.\frac{d}{dt}\right|_{t=0} (R_{\exp(-tX)})_{*\exp(tX)}(Y_L(\exp(tX)) = \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_{*\phi_t(e)}(Y_L(\phi_t(e)) = [X,Y],$$

since $\phi_t(g) = g\exp(tX) = R_{\exp(tX)}(g)$ for every $g \in G$, where R denotes right translation.

As usual, the adjoint representation induces a representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ on the dual of the Lie algebra defined by $\mathrm{Ad}^*_g(a) = a \circ \mathrm{Ad}_{g^{-1}}$, $a \in \mathfrak{g}^*$, which is called the *coadjoint representation* of G.

Example 5.2. Continuing from Example 5.1, we shall describe the coadjoint representation of $SO(3,\mathbb{R})$. The transpose of the linear isomorphism $\widehat{\ }$ induces an isomorphism from $\mathfrak{so}(3,\mathbb{R})^*$ to $(\mathbb{R}^3)^*$ and the latter can be identified naturally with $\mathbb{R}^3$ via the euclidean inner product. The composition of these two isomorphisms gives a way to identify $\mathfrak{so}(3,\mathbb{R})^*$ with $\mathbb{R}^3$ and then, for every v, $w \in \mathbb{R}^3$ we have $\hat{v}^*(\hat{w}) = \langle v, w\rangle$, where $\hat{v}^*$ is the dual of $\hat{v}$ and $\langle , \rangle$ is the euclidean inner product. Now

$$\mathrm{Ad}^*_A(\hat{v}^*)(\hat{w}) = \hat{v}^*(\mathrm{Ad}_{A^{-1}}(\hat{w})) = \langle v, A^{-1}w\rangle = \langle Av, w\rangle,$$

for every $A \in SO(3,\mathbb{R})$, since the transpose of A is A^{-1}. This shows that $\mathrm{Ad}^*_A = A$ via the above identification. Note that the orbit of the point $\hat{v}^* \in \mathfrak{so}(3,\mathbb{R})^* \cong \mathbb{R}^3$ is the set $\{Av : A \in SO(3,\mathbb{R})\}$, which is the sphere of radius $\|v\|$ centered at 0.

The orbit $\mathcal{O}_\mu$ of $\mu \in \mathfrak{g}^*$ under the coadjoint representation is an immersed submanifold of $\mathfrak{g}^*$, since the action is smooth. If G_μ is the isotropy group of μ, then the map $\mathrm{Ad}^*_\cdot(\mu) : G/G_\mu \to \mathcal{O}_\mu$ taking the coset gG_μ to $\mu \circ \mathrm{Ad}_{g^{-1}}$ is a well defined, injective, smooth immersion of the homogeneous space G/G_μ onto $\mathcal{O}_\mu \subset \mathfrak{g}^*$. If the Lie group G is compact, then $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ and the above map an embedding. If however G is not compact, $\mathcal{O}_\mu$ may not be embedded.

Lemma 5.3. *If $\mu \in \mathfrak{g}^*$, then the tangent space of $\mathcal{O}_\mu$ is*

$$T_\mu \mathcal{O}_\mu = \{\mu \circ \mathrm{ad}_X : X \in \mathfrak{g}\}.$$

Proof. Let $\gamma : \mathbb{R} \to G$ be a smooth curve with $\dot{\gamma}(0) = X$. For instance, let $\gamma(t) = \exp(tX)$, in which case $\gamma(t)^{-1} = \exp(-tX)$. Then $\mu(t) = \mu \circ \mathrm{Ad}_{\gamma(t)^{-1}}$ is a smooth curve with values in $\mathcal{O}_\mu \subset \mathfrak{g}^*$ and $\mu(0) = \mu$. If $Y \in \mathfrak{g}$, then $\mu(t)(Y) = \mu(\mathrm{Ad}_{\gamma(t)^{-1}}(Y))$ for every $t \in \mathbb{R}$ and defferentiating at 0 we get

$$\mu'(0)(Y) = \mu(\mathrm{ad}_{(-X)}(Y)) = -\mu(\mathrm{ad}_X(Y)),$$

taking into account the natural identification $T_\mu \mathfrak{g}^* \cong \mathfrak{g}^*$. $\square$

Example 5.4. In the case of the Lie group $SO(3, \mathbb{R})$, for every v, $w \in \mathbb{R}^3 \cong \mathfrak{so}(3, \mathbb{R})$ and $\mu \in \mathbb{R}^3 \cong \mathfrak{so}(3, \mathbb{R})^*$ we have

$$\mu(\mathrm{ad}_{\hat{v}}(\hat{w})) = \langle \mu, v \times w \rangle = \langle \mu \times v, w \rangle.$$

It follows that $T_\mu \mathcal{O}_\mu = \{\mu \times v : v \in \mathbb{R}^3\}$, which is indeed the orthogonal plane to μ, i.e. the tangent plane of the sphere of center 0 and radius $\|\mu\|$ at μ.

The proof of Lemma 5.3 shows that for every $X \in \mathfrak{g}$, the fundamental vector field $X_{\mathfrak{g}^*}$ of the coadjoint action induced by X is given by the formula

$$X_{\mathfrak{g}^*}(\mu) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}(\mu) = -\mu \circ \mathrm{ad}_X.$$

Obviously, $T_\mu \mathcal{O}_\mu = \{X_{\mathfrak{g}^*}(\mu) : X \in \mathfrak{g}\}$. Note that if X, $X' \in \mathfrak{g}$ are such that $X_{\mathfrak{g}^*}(\mu) = X'_{\mathfrak{g}^*}(\mu)$, then

$$-\mu([X, Y]) = X_{\mathfrak{g}^*}(\mu)(Y) = X'_{\mathfrak{g}^*}(\mu)(Y) = -\mu([X', Y])$$

for every $Y \in \mathfrak{g}$. So there is a well defined 2-form ω^- on the coadjoint orbit $\mathcal{O} = \mathcal{O}_\mu$ such that

$$\omega^-_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = -\mu([X, Y])$$

for every $\mu \in \mathcal{O}$ and X, $Y \in \mathfrak{g}$. We call ω^- the *Kirillov 2-form* on $\mathcal{O}$.

The Kirillov 2-form ω^- is non-degenerate, because if $\omega^-_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = 0$ for every $Y_{\mathfrak{g}^*}(\mu) \in T_\mu \mathcal{O}$, then $X_{\mathfrak{g}^*}(\mu)(Y) = -\mu([X, Y]) = 0$ for every $Y \in \mathfrak{g}$. This means $X_{\mathfrak{g}^*}(\mu) = 0$. In order to prove that ω^- is symplectic, it remains to show that it is closed. For this we shall need a series of lemmas.

First note that $\mathrm{Ad}_g[X, Y] = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$ for every X, $Y \in \mathfrak{g}$ and $g \in G$.

Lemma 5.5. $(\mathrm{Ad}_g(X))_{\mathfrak{g}^*} = \mathrm{Ad}^*_g \circ X_{\mathfrak{g}^*} \circ \mathrm{Ad}^*_{g^{-1}}$ *for every $X \in \mathfrak{g}$ and $g \in G$.*

Proof. Let $\gamma : \mathbb{R} \to G$ be a smooth curve $\dot{\gamma}(0) = X$. For instance $\gamma(t) = \exp(tX)$, and then

$$\mathrm{Ad}_g(X) = \left.\frac{d}{dt}\right|_{t=0} g\gamma(t)g^{-1}.$$

Therefore,

$$(\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\mu) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{g\gamma(t)g^{-1}}(\mu) = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_{\gamma(t)} \circ \mathrm{Ad}^*_{g^{-1}})(\mu)$$

$$= (\mathrm{Ad}^*_g \circ X_{\mathfrak{g}^*} \circ \mathrm{Ad}^*_{g^{-1}})(\mu). \quad \square$$

Lemma 5.6. *The Kirillov 2-form is* Ad^**-invariant.*

Proof. Let $\mu \in \mathfrak{g}^*$ and $\nu = \mathrm{Ad}^*_g(\mu)$, $g \in G$. By Lemma 5.5,

$$(\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\nu) = \mathrm{Ad}^*_g(X_{\mathfrak{g}^*}(\mu)).$$

Thus, for every X, $Y \in \mathfrak{g}$ we have

$$((\mathrm{Ad}^*_g)^*\omega^-)_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)) = \omega^-_\nu((\mathrm{Ad}_g(X))_{\mathfrak{g}^*}(\nu), (\mathrm{Ad}_g(Y))_{\mathfrak{g}^*}(\nu))$$

$$= -\nu([\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]) = -\nu(\mathrm{Ad}_g[X,Y]) = -\mu([X,Y])$$

$$= \omega^-_\mu(X_{\mathfrak{g}^*}(\mu), Y_{\mathfrak{g}^*}(\mu)). \quad \square$$

For every $\nu \in \mathfrak{g}^*$ we have a well defined 1-form ν_L on G such that

$$(\nu_L)_g = \nu \circ (L_{g^{-1}})_{*g} \in T^*_g G.$$

Moreover, ν_L is left invariant, because for every $h \in G$ we have

$$(L^*_h \nu_L)_g = (\nu_L)_{L_h(g)} \circ (L_h)_{*g} = \nu \circ ((L_{g^{-1}h^{-1}})_{*hg} \circ (L_h)_{*g})$$

$$= \nu \circ (L_{g^{-1}h^{-1}} \circ L_h)_{*g} = \nu_L(g).$$

Obviously, $i_{X_L}\nu_L$ is constant and equal to $\nu(X)$ for every $X \in \mathfrak{g}$.

Let $\nu \in \mathcal{O}$ and $\phi_\nu : G \to \mathcal{O}$ be the submersion $\phi_\nu(g) = \mathrm{Ad}^*_g(\nu)$. The 2-form $\sigma = \phi^*_\nu\omega^-$ on G is left invariant, because

$$L^*_g\sigma = (\phi_\nu \circ L_g)^*\omega^- = (\mathrm{Ad}^*_g \circ \phi_\nu)^*\omega^- = \phi^*_\nu((\mathrm{Ad}^*_g)^*\omega^-) = \phi^*_\nu\omega^- = \sigma$$

for every $g \in G$, since ω^- is Ad^*-invariant and $\phi_\nu \circ L_g = \mathrm{Ad}^*_g \circ \phi_\nu$.

Lemma 5.7. *For every* X*,* $Y \in \mathfrak{g}$ *we have* $\sigma(X_L, Y_L) = -\nu_L([X_L, Y_L])$*.*

Proof. First we observe that

$$(\phi^*_\nu\omega^-)_e(X,Y) = \omega^-_\nu((\phi_\nu)_{*e}(X), (\phi_\nu)_{*e}(Y)) = \omega^-_\nu(X_{\mathfrak{g}^*}(\nu), Y_{\mathfrak{g}^*}(\nu)) = -\nu([X,Y]).$$

Therefore,

$$\sigma(X_L, Y_L)(e) = (\phi^*_\nu\omega^-)_e(X,Y) = -\nu([X,Y]) = -\nu_L([X_L, Y_L])(e).$$

Since the smooth functions $\sigma(X_L, Y_L)$, $-\nu_L([X_L, Y_L]) : G \to \mathbb{R}$ are left invariant and take the same value at e, they must be identical. $\square$

Note that

$$(d\nu_L)(X_L, Y_L) = X_L(\nu_L(Y_L)) - Y_L(\nu_L(X_L)) - \nu_L([X_L, Y_L]) = -\nu_L([X_L, Y_L]),$$

since the functions $\nu_L(Y_L) = i_{Y_L}\nu_L$ and $\nu_L(X_L) = i_{X_L}\nu_L$ are constant.

Lemma 5.8. *The 2-form σ is exact and $\sigma = d\nu_L$.*

Proof. Since σ is left invariant, for any two smooth vector fields X, Y on G we have

$$\begin{aligned}
\sigma(X, Y)(g) &= (L^*_{g^{-1}}\sigma)_g(X(g), Y(g)) = \sigma_e((L_{g^{-1}})_{*g}(X(g)), (L_{g^{-1}})_{*g}(Y(g))) \\
&= \sigma(X'_L, Y'_L)(e) \qquad \text{(setting } X' = (L_{g^{-1}})_{*g}(X(g)) \text{ and similarly for } Y'\text{)} \\
&= (d\nu_L)(X'_L, Y'_L)(e) \qquad \text{(by Lemma 5.7)} \\
&= (d\nu_L)_g((L_g)_{*e}(X'), (L_g)_{*e}(Y')) \qquad \text{(since } \nu_L \text{ is left invariant)} \\
&= (d\nu_L)_g(X(g), Y(g)) = (d\nu_L)(X, Y)(g). \qquad \square
\end{aligned}$$

Proposition 5.9. *The Kirillov 2-form ω^- on $\mathcal{O}$ is closed and therefore symplectic.*

Proof. By Lemma 5.8, $d(\phi^*_\nu\omega^-) = d\sigma = d(d\nu_L)) = 0$. Hence $\phi^*_\nu(d\omega^-) = 0$. But ϕ^*_ν is injective, since ϕ_ν is a submersion. It follows that $d\omega^- = 0$. $\square$

Corollary 5.10. *Every orbit of the coadjoint action of a Lie group G on its dual Lie algebra $\mathfrak{g}^*$ has even dimension.* $\square$

We shall end this section with a couple of illustrating examples.

Example 5.11. As we saw in Example 5.2, if $\mu \in \mathfrak{so}(3, \mathbb{R})^* \cong \mathbb{R}^3$, then $\mathcal{O}_\mu$ is the sphere centered at 0 with radius $\|\mu\|$. Let v, $w \in \mathfrak{so}(3, \mathbb{R}) \cong \mathbb{R}^3$. Then $v_{\mathbb{R}^3} = \mu \times v \in T_\mu\mathcal{O}_\mu$ and $w_{\mathbb{R}^3} = \mu \times w \in T_\mu\mathcal{O}_\mu$. Hence the Kirillov 2-form on $\mathcal{O}_\mu$ is given by the formula

$$\omega^-_\mu(v_{\mathbb{R}^3}, w_{\mathbb{R}^3}) = -\langle \mu, v \times w\rangle.$$

Since $\mathcal{O}_\mu$ is a sphere, its area element is given by the formula

$$dA(v, w) = \langle N, v \times w\rangle,$$

where N is the outer unit normal vector. It follows that

$$\begin{aligned}
dA(\mu \times v, \mu \times w) &= \langle \frac{1}{\|\mu\|}\mu, (\mu \times v) \times (\mu \times w)\rangle = \langle \frac{1}{\|\mu\|}\mu, \langle \mu, \mu \times w\rangle v - \langle v, \mu \times w\rangle \mu\rangle \\
&= -\|\mu\|\langle v, \mu \times w\rangle = \|\mu\|\langle \mu, v \times w\rangle,
\end{aligned}$$

where we have used the property $(a \times b) \times c = \langle a, c\rangle b - \langle b, c\rangle a$ of the vector product in $\mathbb{R}^3$. This shows that

$$\omega^- = -\frac{1}{\|\mu\|}dA.$$

Example 5.12. The connected Lie group of the orientation preserving affine transformations of $\mathbb{R}$ is represented as a group of matrices by

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R}\}.$$

Its Lie algebra is

$$\mathfrak{g} = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}\} \cong \mathbb{R}^2$$

with Lie bracket $[A, B] = AB - BA$. The exponential map is computed as follows. Let x, $y \in \mathbb{R}$ with $x \neq 0$. Let $\gamma : \mathbb{R} \to G$ be the smooth curve defined by

$$\gamma(t) = \begin{pmatrix} e^{tx} & \frac{y}{x}(e^{tx} - 1) \\ 0 & 1 \end{pmatrix}.$$

Then $\gamma(0) = I_2$ and

$$\gamma'(t) = \begin{pmatrix} xe^{tx} & ye^{tx} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{tx} & \frac{y}{x}(e^{tx} - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = (L_{\gamma(t)})_{*I_2} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}.$$

This shows that if

$$W = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix},$$

then $\exp(tW) = \gamma(t)$. If now

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

then

$$\mathrm{Ad}_A(W) = \frac{d}{dt}\bigg|_{t=0} A\gamma(t)A^{-1} = AWA^{-1} = \begin{pmatrix} x & ay - bx \\ 0 & 0 \end{pmatrix}.$$

So the orbit of W under the adjoint action is the line

$$\{\begin{pmatrix} x & t \\ 0 & 0 \end{pmatrix} : t \in \mathbb{R}\},$$

when $x \neq 0$. If $x = 0$ and $y > 0$, it is the upper half-line, and if $y < 0$ it is the lower half-line. We see now that the adjoint representation cannot be equivalent (in any sense) to the coadjoint representation, since the orbits of the latter have even dimension. In the sequel, we shall find the orbits of the coadjoint representation and the Kirillov 2-form.

Let $\mu \in \mathfrak{g}^* \cong \mathbb{R}^2$, the isomorphism given by euclidean inner product. This means that if $\mu = (\alpha^*, \beta^*)$, then $\mu(W) = \alpha^* x + \beta^* y$. So,

$$\mathrm{Ad}^*_{A^{-1}}(\mu)(W) = \mu(\mathrm{Ad}_A(W)) = \alpha^* x + \beta^*(ay - bx) = (\alpha^* - \beta^* b)x + \beta^* ay$$

which implies that $\mathrm{Ad}^*_{A^{-1}}(\alpha^*, \beta^*) = (\alpha^* - \beta^* b, \beta^* a)$. Therefore,

$$\mathcal{O}_\mu = \{(\alpha^* - \beta^* b, \beta^* a) : a > 0, b \in \mathbb{R}\}$$

for $\mu = (\alpha^*, \beta^*)$, as before.

If $\beta^* \neq 0$, then for $\beta^* > 0$ the coadjoint orbit $\mathcal{O}_\mu$ is the open upper half plane and for $\beta^* < 0$ it is the open lower half plane. For $\beta^* = 0$ we have $\mathcal{O}_\mu = \{(\alpha^*, 0)\}$. In the case $\beta^* \neq 0$, the Kirillov 2-form ω^- on $\mathcal{O}_\mu$ satisfies

$$\omega_\mu^-((W_1)_{\mathfrak{g}^*}, (W_2)_{\mathfrak{g}^*}) = -\mu([W_1, W_2]) = -\mu \begin{pmatrix} 0 \\ x_1y_2 - x_2y_1 \end{pmatrix} = \beta^*(x_2y_1 - x_1y_2),$$

where

$$W_j = \begin{pmatrix} x_j & y_j \\ 0 & 0 \end{pmatrix}, \quad j = 1, 2,$$

since

$$[W_1, W_2] = \begin{pmatrix} 0 & x_1y_2 - x_2y_1 \\ 0 & 0 \end{pmatrix}.$$

Consequently, $\omega^- = -\beta^* d\alpha^* \wedge d\beta^*$.

2.6 Homogeneous symplectic manifolds

Let $\mathfrak{g}$ be a (real) Lie algebra and let $\Lambda^k(\mathfrak{g})$, $k \geq 0$, denote the vector space of all skew-symmetric covariant k-tensors on $\mathfrak{g}$. For every $k \geq 0$, let $\delta : \Lambda^k(\mathfrak{g}) \to \Lambda^{k+1}(\mathfrak{g})$ be the linear map defined by

$$(\delta\omega)(X_0, X_1, ..., X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).$$

For $k = 0$, we have $\delta = 0$, and for $k = 1$ we have

$$(\delta\omega)(X_0, X_1) = -\omega([X_0, X_1]).$$

A standard computation shows that $\delta \circ \delta = 0$. Let $Z^k(\mathfrak{g}) = \Lambda^k(\mathfrak{g}) \cap \mathrm{Ker}\delta$ and $B^k(\mathfrak{g}) = \Lambda^k(\mathfrak{g}) \cap \mathrm{Im}\delta$. The quotient $H^k(\mathfrak{g}) = Z^k(\mathfrak{g})/B^k(\mathfrak{g})$ is called the Lie algebra k-cohomology of $\mathfrak{g}$. Obviously, $H^0(\mathfrak{g}) = \{0\}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then $\Lambda^1(\mathfrak{g}) = \mathfrak{g}^*$. For every $k \geq 0$ the space $\Lambda^k(\mathfrak{g})$ can be identified in the obvious way with the vector space of left invariant differential k-forms on G. The adjoint representation induces a left action Ad^* on $\Lambda^k(\mathfrak{g})$, which for $k = 1$ is just the coadjoint representation. So $\mathrm{Ad}_g^*(\theta) = (\mathrm{Ad}_{g^{-1}})^*\theta$, for $\theta \in \Lambda^k(\mathfrak{g})$. It is evident that $Z^k(\mathfrak{g})$ is an Ad^*-invariant subspace.

Lemma 6.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H^1(\mathfrak{g}) = \{0\}$ and $H^2(\mathfrak{g}) = \{0\}$, then the* Ad^**- actions of G on $\mathfrak{g}^*$ and $Z^2(\mathfrak{g})$ are isomorphic.*

Proof. Since $H^2(\mathfrak{g}) = \{0\}$, for every $\theta \in Z^2(\mathfrak{g})$ there exists $\mu \in \mathfrak{g}^*$ such that $\delta\mu = \theta$. On the other hand, since $H^1(\mathfrak{g}) = \{0\}$, we have $Z^1(\mathfrak{g}) = B^1(\mathfrak{g}) = \delta(\Lambda^0(\mathfrak{g})) = 0$. Thus, if $\delta\mu = 0$, then $\mu = 0$. It follows that $\delta : \mathfrak{g}^* \to Z^2(\mathfrak{g})$ is an isomorphism. It is obvious that δ is Ad^*-equivariant. □

Let now (M,ω) be a symplectic manifold, G a Lie group and $\phi : G \times M \to M$ a smooth, symplectic action. Let $\phi_g = \phi(g,.)$ and $\phi^p = \phi(.,p)$ for $g \in G$ and $p \in M$. Then, $\phi_g \circ \phi^p = \phi^p \circ L_g$ and $\phi^{\phi_g(p)} = \phi^p \circ R_g$. The closed 2-form $(\phi^p)^*\omega$ on G is left invariant, because

$$(L_g)^*((\phi^p)^*\omega) = (\phi^p \circ L_g)^*\omega = (\phi_g \circ \phi^p)^*\omega = (\phi^p)^*((\phi_g)^*\omega) = (\phi^p)^*\omega,$$

since ϕ_g is a symplectomorphism. Let $\Psi : M \to Z^2(\mathfrak{g})$ be the smooth map defined by

$$\Psi(p) = ((\phi^p)^*\omega)_e.$$

Since $(\phi^p)^*\omega$ is left invariant, Ψ is equivariant. Indeed,

$$((\phi^{\phi_g(p)})^*\omega)_e = ((\phi^p \circ R_g)^*\omega)_e = (R_g^*((\phi^p)^*\omega))_e = (\mathrm{Ad}_{g^{-1}})^*((\phi^p)^*\omega)_e.$$

In case the action is transitive, then $\Psi(M)$ is precisely one orbit in $Z^2(\mathfrak{g})$. Recall that if the action is transitive, then M is diffeomorphic to the homogeneous space G/H, where H is the isotropy group of any point of M.

Proposition 6.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H^1(\mathfrak{g}) = \{0\}$ and $H^2(\mathfrak{g}) = \{0\}$, then for every $\theta \in Z^2(\mathfrak{g})$ there exists a homogenous symplectic G-manifold M such that $\Psi(M) = \mathcal{O}_\theta$, where $\mathcal{O}_\theta$ is the orbit of θ under the* Ad^**-action of G on $Z^2(\mathfrak{g})$.*

Proof. Let θ_L denote the left invariant 2-form on G defined by θ. According to Lemma 6.1, there exists a unique $\mu \in \mathfrak{g}^*$ such that $\delta\mu = \theta$. Let G_μ be the isotropy group of μ under the coadjoint representation. Then,

$$G_\mu = \{g \in G : \mathrm{Ad}_g^*\mu = \mu\} = \{g \in G : (L_{g^{-1}} \circ R_g)^*\mu_L = \mu_L\}$$

$$= \{g \in G : (R_g)^*\mu_L = \mu_L\}.$$

If $X \in \mathfrak{g}$, the flow $(\psi_t)_{t\in\mathbb{R}}$ of X_L is given by the formula

$$\psi_t(g) = g\exp(tX) = R_{\exp(tX)}(g).$$

It follows that the Lie algebra of G_μ is $\mathfrak{g}_\mu = \{X \in \mathfrak{g} : L_{X_L}\mu_L = 0\}$. But $i_{X_L}\mu_L$ is a constant function, and therefore $L_{X_L}\mu_L = i_{X_L}(d\mu_L) = i_{X_L}\theta_L$. So,

$$\mathfrak{g}_\mu = \{X \in \mathfrak{g} : i_{X_L}\theta_L = 0\}.$$

Let H_θ be the connected component of G_μ wich contains e. The Lie algebra of H_θ is $\mathfrak{g}_\mu$, and of course H_θ is a closed Lie subgroup of G. Recall that the homogeneous space $M = G/H_\theta$ becomes a smooth manifold in a unique way such that the quotient map $\pi : G \to M$ is smooth and it has local cross sections. In particular π is a submersion.

Every $g \in G$ is contained in the domain U of a chart $(U, x^1, x^2, ..., x^m)$ of G, where $m = \dim G$, which maps U diffeomorphically onto $\mathbb{R}^m$, such that

$(\pi(U), x^1, x^2, ..., x^n)$ is a chart on M, where $m-n = \dim H_\theta$, and in these local coordinates $\pi(x^1, x^2, ..., x^m) = (x^1, x^2, ..., x^n)$. There are smooth functions $a_{ij} : U \to \mathbb{R}$, $1 \leq i < j \leq m$, such that

$$\theta_L|_U = \sum_{1\leq i<j\leq m} a_{ij} dx^i \wedge dx^j.$$

The tangent space of the submanifold gH_θ of G at g has basis $\{\frac{\partial}{\partial x^{n+1}}, ..., \frac{\partial}{\partial x^m}\}$. Since $\tilde{\theta}_L(\frac{\partial}{\partial x^l}) = 0$ for $n < l \leq m$, it follows that

$$\sum_{l<j} a_{lj} dx^j - \sum_{i<l} a_{il} dx^i = 0.$$

Therefore $a_{ij} = 0$, when $i > n$ or $j > n$, which means that

$$\theta_L|_U = \sum_{1\leq i<j\leq n} a_{ij} dx^i \wedge dx^j$$

and then

$$0 = d(\theta_L) = \sum_{1\leq i<j\leq n} \left(\sum_{l=1}^{m} \frac{\partial a_{ij}}{\partial x^l} dx^l \right) \wedge dx^i \wedge dx^j.$$

Hence $\frac{\partial a_{ij}}{\partial x^l} = 0$ for $n < l \leq m$, and so the functions a_{ij} do not depend on the coordinates $x^{n+1}, ..., x^m$. This implies that θ_L descends to a well defined 2-form $\bar{\theta}_L$ on $\pi(U)$ given by the same formula. It is standard and easy, but somewhat tedious, to show that there is a well defined closed 2-form ω on M such that $\pi^*\omega = \theta_L$ and $\omega|_{\pi(U)} = \bar{\theta}_L$. Moreover, ω is non-degenerate, because $\tilde{\omega}_{\pi(g)}(\pi_{*g}(v)) = 0$ if and only if $(\tilde{\theta}_L)_g(v) = 0$ or equivalently $v \in T_g(gH_\theta)$, that is $\pi_*(v) = 0$. So we have so far shown that (M, ω) is a symplectic manifold.

Let $\phi : G \times M \to M$ be the natural transitive left action of G on M so that $\phi_g(hH_\theta) = (gh)H_\theta$, $g \in G$. Then $\phi_g \circ \pi = \pi \circ L_g$ and therefore

$$\pi^*(\phi_g^*\omega) = (\phi_g \circ \pi)^*\omega = (\pi \circ L_g)^*\omega = L_g^*(\pi^*\omega) = L_g^*\theta_L = \theta_L = \pi^*\omega.$$

Since π is a submersion, π^* is injective in the level of forms, and hence $\phi_g^*\omega = \omega$. This shows that the action is symplectic. In order to complete the proof, it remains to show that $\Psi(M) = \mathcal{O}_\theta$. If $p = gH_\theta \in M$, then $\phi^p = \pi \circ R_g$, and for every X, $Y \in \mathfrak{g}$ we have

$$\Psi(p)(X, Y) = ((\phi^p)^*\omega)_e(X, Y) = \omega_{\pi(g)}(\pi_{*g}((R_g)_{*e}(X)), \pi_{*g}((R_g)_{*e}(Y)))$$

$$= (\pi^*\omega)_g((R_g)_{*e}(X), (R_g)_{*e}(Y)) = (\theta_L)_g((R_g)_{*e}(X), (R_g)_{*e}(Y))$$

$$= \theta((L_{g^{-1}} \circ R_g)_{*e}(X), (L_{g^{-1}} \circ R_g)_{*e}(X)) = \mathrm{Ad}_g^*(\theta)(X, Y).$$

In other words $\Psi(gH_\theta) = \mathrm{Ad}_g^*(\theta)$ for every $g \in G$ and therefore $\Psi(M) = \mathcal{O}_\theta$. □

Let again $\theta \in Z^2(\mathfrak{g})$ and suppose that (M, ω) is a symplectic manifold, on which the Lie group G with Lie algebra $\mathfrak{g}$ acts transitively, symplectically and such that $\Psi(M) = \mathcal{O}_\theta$. Then M is diffeomorphic to the homogeneous space G/H, where H is the isotropy group of any point of M and necessarily $\theta_L = (\phi^p)^*\omega$, where $p \in M$. The Lie algebra of H is

$$\mathfrak{h} = \{X \in \mathfrak{g} : i_X\theta = 0\},$$

If H_θ is the connected component of H which contains e, then $\Psi(G/H_\theta) = \mathcal{O}_\theta$, as the proof of Proposition 6.2 shows. The homogeneous space G/H_θ is a covering space of M. These show that if M amd N are two homogeneous, symplectic G-manifolds with $\Psi(M) = \Psi(N)$, then N is a covering space of M or vice versa.

Summarizing the results of this section, we have proved the following.

Theorem 6.3. (Kostant-Souriau) *Let G be a Lie group with Lie algebra $\mathfrak{g}$ such that $H^1(\mathfrak{g}) = \{0\}$ and $H^2(\mathfrak{g}) = \{0\}$. Then, up to covering spaces, the homogeneous, symplectic G-manifolds are in one-to-one, onto correspondence with the coadjoint orbits in $\mathfrak{g}^*$.* $\square$

Before we end this section, we need to make some remarks about the assumptions in the Kostant-Souriau theorem. If G is a compact, connected Lie group with Lie algebra $\mathfrak{g}$, then $H^k(\mathfrak{g})$ is isomorphic to the k-th deRham cohomology, and so to the k-th real singular cohomology $H^k(G;\mathbb{R})$ of G for every $k \geq 0$. Moreover, in this case the condition $H^1(G;\mathbb{R}) = 0$ implies that $H^2(G;\mathbb{R}) = 0$ also. For example, the special orthogonal group $SO(3,\mathbb{R})$ is a compact, connected Lie group and is diffeomorphic to the 3-dimensional real projective space $\mathbb{R}P^3$. Therefore, its Lie algebra $\mathfrak{so}(3,\mathbb{R})$ satisfies the assumptions of the Kostant-Souriau theorem.

2.7 Poisson manifolds

In this section we shall describe an algebraic foundation of mechanics. A *Poisson algebra* is a triple $(\mathcal{A}, \{,\}, \cdot)$, where the pair $(\mathcal{A}, \{,\})$ is Lie algebra, while at the same time $\mathcal{A}$ is a commutative ring with a unit element and multiplication $\cdot$, such that we have a Leibniz formula

$$\{f, g \cdot h\} = h \cdot \{f, g\} + g \cdot \{f, h\}$$

for every f, g, $h \in \mathcal{A}$. From section of 2.4 follows that if (M, ω) is a symplectic manifold, then $(C^\infty(M), \{,\}, \cdot)$ is a Poisson algebra, where $\{,\}$ is the Poisson bracket with respect to ω and $\cdot$ is the usual multiplication of functions. A map $\phi : \mathcal{A} \to \mathcal{B}$ of Poisson algebras is called a homomorphism if is a Lie algebra homomorphism and a homomorphism of commutative rings with unit element.

The Leibniz formula says that for every $f \in \mathcal{A}$ the linear map $\mathrm{ad}_f : \mathcal{A} \to \mathcal{A}$ with $\mathrm{ad}_f(g) = \{g, f\}$ is a derivation. It is called the *Hamiltonian derivation* defined by f. An $f \in \mathcal{A}$ is called a *Casimir element* if $\{f, g\} = 0$ for every $g \in \mathcal{A}$. For example, the unit $1 \in \mathcal{A}$ is a Casimir element, since

$$\{f, 1\} = \{f, 1 \cdot 1\} = 1 \cdot \{f, 1\} + 1 \cdot \{f, 1\} = 2\{f, 1\} = 0$$

for every $f \in \mathcal{A}$. A Poisson algebra $\mathcal{A}$ is called non-degenerate if every Casimir element of $\mathcal{A}$ is of the form $t \cdot 1$, $t \in \mathbb{R}$.

A *Poisson manifold* is a smooth manifold M together with a Poisson structure on the ring of smooth functions $C^\infty(M)$. So the Poisson structure on M is completely determined by the Lie-Poisson bracket $\{,\}$ on $C^\infty(M)$. If $(U, x^1, x^2, ..., x^n)$ is a chart on M, since ad_f is a derivation of $C^\infty(M)$, it is a smooth vector field on M. So,

$$\mathrm{ad}_f|_U = \sum_{k=1}^{n} \{x^k, f\} \frac{\partial}{\partial x^k}.$$

For every f, $g \in C^\infty(M)$ we have

$$\{g, f\}|_U = \sum_{k=1}^{n} \{x^k, f\} \frac{\partial g}{\partial x^k} = -\sum_{k=1}^{n} \{f, x^k\} \frac{\partial g}{\partial x^k} = \sum_{j,k=1}^{n} \{x^k, x^j\} \frac{\partial f}{\partial x^j} \cdot \frac{\partial g}{\partial x^k}.$$

It follows that the Poisson structure on M is determined by a contravariant, skew-symmetric 2-tensor W, which is called the structural tensor of the Poisson structure. For every $p \in M$, the skew-symmetric, bilinear form $W_p : T_p^*M \times T_p^*M \to \mathbb{R}$ is determined by the structural matrix $(\{x^j, x^k\})_{1 \le j,k \le n}$. Its rank is called the rank of the Poisson structure at p.

Proposition 7.1. *The Poisson structure of a Poisson manifold M is defined by a symplectic structure on M if and only if the structural matrix is invertible at every point of M.*

Proof. Let (M, ω) be a symplectic manifold and $\{,\}$ be the corresponding Poisson bracket. Then the Poisson tensor is given by $W(df, dg) = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields with Hamiltonian functions f, $g \in C^\infty(M)$, respectively. Let f be such that $W(df, dg) = 0$ for every $g \in C^\infty(M)$. Since T_p^*M is generated by $\{(dg)(p) : g \in C^\infty(M)\}$ for every $p \in M$ and ω is non-degenerate, T_pM is generated by $\{X_g(p) : g \in C^\infty(M)\}$. It follows now that $X_f(p) = 0$ for every $p \in M$. Therefore, $df = 0$ on M. This shows that the structural matrix is invertible. If M is connected, the Poisson structure is also non-degenerate. For the converse, let M be a Poisson manifold, such that the structural matrix W is everywhere invertible. For $f \in C^\infty(M)$ put $X_f = \mathrm{ad}_f$. We define

$$\omega(X_f, X_g) = \{f, g\} = W(df, dg) = df(X_g).$$

Since T_p^*M is generated by $\{(dg)(p) : g \in C^\infty(M)\}$ and W is invertible, it follows that ω is a non-degenerate 2-form and it remains to show that ω is closed. For this, we observe first that

$$\begin{aligned}[X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) = X_f(\{h, g\}) - X_g(\{h, f\}) \\ &= \{\{h, g\}, f\} - \{\{h, f\}, g\} = -\{h, \{f, g\}\} = -X_{\{f,g\}}(h)\end{aligned}$$

for every $h \in C^\infty(M)$. Consequently,

$$d\omega(X_f, X_g, X_h) = X_f(\omega(X_g, X_h)) - X_g(\omega(X_f, X_h)) + X_h(\omega(X_f, X_g))$$

$$-\omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f)$$
$$= 2[\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}] = 0. \quad \square$$

Example 7.2. Let $(\mathfrak{g}, [,])$ be a (real) Lie algebra of finite dimension n and $\mathfrak{g}^*$ be its dual. Since $\mathfrak{g}$ has finite demension, the double dual $\mathfrak{g}^{**}$ is naturally isomorphic to $\mathfrak{g}$, and so their elements can be identified. For f, $g \in C^\infty(\mathfrak{g}^*)$ let $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ be defined by

$$\{f, g\}(\mu) = \mu[df(\mu), dg(\mu)]$$

for $\mu \in \mathfrak{g}^*$. It is obvious that the bracket $\{,\}$ is bilinear and skew-symmetric. Moreover, the Leibniz formula holds, since it holds for d. In order to have a Poisson manifold, it remains to verify the Jacobi identity. If $\{x_1, x_2, ..., x_n\}$ is a basis of $\mathfrak{g}$, then x_1, x_2,...,x_n can be considered as (global) coordinate functions on $\mathfrak{g}$. If f, $g \in C^\infty(\mathfrak{g}^*)$, then

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j}.$$

Since $\{x_i, x_j\}(\mu) = \mu[dx_i(\mu), dx_j(\mu)] = \mu[x_i, x_j]$ for every $\mu \in \mathfrak{g}^*$, it follows from the Jacobi identity on $\mathfrak{g}$, that it also holds for $\{,\}$ on the the set $\{x_1, x_2, ..., x_n\}$. In general, if f, g, $h \in C^\infty(\mathfrak{g}^*)$ note first that

$$\sum_{k=1}^{n} \{x_k, x_i\}\{\frac{\partial f}{\partial x_k}, x_j\} = \sum_{k,l=1}^{n} \{x_k, x_i\}\{x_l, x_j\} \frac{\partial^2 f}{\partial x_l \partial x_k}$$

$$= \sum_{k,l=1}^{n} \{x_l, x_j\}\{x_k, x_i\} \frac{\partial^2 f}{\partial x_k \partial x_l} = \sum_{k=1}^{n} \{x_k, x_j\}\{\frac{\partial f}{\partial x_k}, x_i\}.$$

Now we compute

$$\{\{f, g\}, h\} = \sum_{i,j,k=1}^{n} \{\{x_i, x_j\}, x_k\} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k}$$

$$+ \sum_{i,j,k=1}^{n} \{x_i, x_j\}\{\frac{\partial f}{\partial x_i}, x_k\} \frac{\partial h}{\partial x_k} \cdot \frac{\partial g}{\partial x_j} + \sum_{i,j,k=1}^{n} \{x_i, x_j\}\{\frac{\partial g}{\partial x_j}, x_k\} \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_i}.$$

Similarly,

$$\{\{g, h\}, f\} = \sum_{i,j,k=1}^{n} \{\{x_j, x_k\}, x_i\} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k}$$

$$+ \sum_{i,j,k=1}^{n} \{x_j, x_k\}\{\frac{\partial g}{\partial x_j}, x_i\} \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j,k=1}^{n} \{x_j, x_k\}\{\frac{\partial h}{\partial x_k}, x_i\} \frac{\partial g}{\partial x_j} \cdot \frac{\partial f}{\partial x_i}$$

and

$$\{\{h, f\}, g\} = \sum_{i,j,k=1}^{n} \{\{x_k, x_i\}, x_j\} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k}$$

$$+\sum_{i,j,k=1}^{n}\{x_k,x_i\}\{\frac{\partial h}{\partial x_k},x_j\}\frac{\partial g}{\partial x_j}\cdot\frac{\partial f}{\partial x_i}+\sum_{i,j,k=1}^{n}\{x_k,x_i\}\{\frac{\partial f}{\partial x_i},x_j\}\frac{\partial h}{\partial x_k}\cdot\frac{\partial g}{\partial x_j}.$$

Summing up we get

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=$$

$$\sum_{i,j,k=1}^{n}\Big(\{\{x_i,x_j\},x_k\}+\{\{x_j,x_k\},x_i\}+\{\{x_k,x_i\},x_j\}\Big)\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial h}{\partial x_k}=0.$$

In this way $\mathfrak{g}^*$ becomes a Poisson manifold.

If M_1 and M_2 are two Poisson manifolds, a smooth map $h: M_1\to M_2$ is called *Poisson* if $h^*: C^\infty(M_2)\to C^\infty(M_1)$ is a homomorphism of Poisson algebras.

Let M be a Poisson manifold. For every $f\in C^\infty(M)$, the smooth vector field X_f corresponding to the Hamiltonian derivation $\mathrm{ad}_f=\{.,f\}$ is called the *Hamiltonian vector field* of f. This definition agrees with the definition of section 4 in case M is symplectic.

Proposition 7.3. *Let M be a Poisson manifold and X_f be a Hamiltonian vector field on M with Hamiltonian function $f\in C^\infty(M)$. Let $\phi: D\to M$ be the flow of X_f, where $D\subset\mathbb{R}\times M$ is an open neighbourhood of $\{0\}\times M$.*

(i) If $g\in C^\infty(M)$, then

$$\frac{d}{dt}(g\circ\phi_t)=\{g,f\}\circ\phi_t=\{g\circ\phi_t,f\}.$$

(ii) $f\circ\phi_t=f$

(iii) The flow of the Hamiltonian vector field X_f consists of Poisson maps.

Proof. (i) If $p\in M$, then on the one hand

$$\frac{d}{dt}(g\circ\phi_t)=X_f(g)(\phi_t(p))=\{g,f\}(\phi_t(p))$$

and on the other hand

$$\frac{d}{dt}(g\circ\phi_t)=g_{*\phi_t(p)}((\phi_t)_{*p}(X_f(p)))=(g\circ\phi_t)_{*p}(X_f(p))=X_f(g\circ\phi_t)(p)=\{g\circ\phi_t,f\}.$$

(ii) This is obvious from (i) taking $g=f$.

(iii) Let g_1, $g_2\in C^\infty(M)$ and let $g\in C^\infty(D)$ be defined by

$$g(t,p)=\{g_1\circ\phi_t,g_2\circ\phi_t\}(p)-\{g_1,g_2\}(\phi_t(p)).$$

From (i) and the Jacobi identity we have

$$\frac{\partial g}{\partial t}=\{\frac{d}{dt}(g_1\circ\phi_t),g_2\circ\phi_t\}(p)+\{g_1\circ\phi_t,\frac{d}{dt}(g_2\circ\phi_t)\}(p)-\frac{d}{dt}\{g_1,g_2\}(\phi_t(p))=$$

$$\{\{g_1\circ\phi_t,f\},g_2\circ\phi_t\}+\{g_1\circ\phi_t,\{g_2\circ\phi_t,f\}\}-\{\{g_1,g_2\}\circ\phi_t,f\}=$$

$$\{g_t, f\} = X_f(g_t),$$

where as usual $g_t = g(t, .)$, and $g(0, p) = 0$. By uniqueness of solutions of ordinary differential equations, we must necessarily have $g(t, p) = 0$ for all t such that $(t, p) \in D$. □

If $h : M_1 \to M_2$ is a Poisson map of Poisson manifolds and $f \in C^\infty(M_2)$, then $h_{*p}(X_{h^*(f)}(p)) = X_f(h(p))$ for every $p \in M_1$. Therefore, h transforms integral curves of $X_{h^*(f)}$ in M_1 to integral curves of X_f on M_2.

If M is a Poisson manifold and $N \subset M$ is an immersed submanifold, then N is called a *Poisson submanifold* if the inclusion $i : N \hookrightarrow M$ is a Poisson map. On every Poisson manifold M one can define an equivalence relation $\sim$ by setting $p \sim q$ if and only if there is a piecewise smooth curve from p to q whose smooth parts are pieces of integral curves of Hamiltonian vector fields of M. The equivalence classes are called the *symplectic leaves* of the Poisson structure of M. We shall prove that the symplectic leaves are immersed submanifolds and carry a unique symplectic structure so that the become Poisson submanifolds of M.

Let $p \in M$ and f_1, f_2,...,$f_k \in C^\infty(M)$ be such that the set $\{X_{f_1}(p), ..., X_{f_k}(p)\}$ is a basis of $\mathrm{Im}\tilde{W}_p$, where W_p is the Poisson tensor and $\tilde{W}_p : T_p^*M \to T_p^{**}M \cong T_pM$ is the induced linear map. In other words, $X_{f_j}(p) = \tilde{W}_p(df_j(p))$. There exists some $\epsilon > 0$ and an open neighbourhood U of p such that the flow ϕ_j of X_{f_j} is defined on $(-\epsilon, \epsilon) \times U$ for every $1 \le j \le k$. Taking a smaller $\epsilon > 0$, we may assume that

$$\Phi_p(t_1, t_2, ..., t_k) = (\phi_{1,t_1} \circ \phi_{2,t_2} \circ ... \circ \phi_{k,t_k})(p)$$

is defined for $|t_j| < \epsilon$, $1 \le j \le k$. Obviously, Φ_p is smooth and

$$(\Phi_p)_{*0}(\frac{\partial}{\partial t_j}) = X_{f_j}(p)$$

for $1 \le j \le k$. So, $(\Phi_p)_{*0}$ is a monomorphism and from the inverse function theorem there exists an open neighbourhood V_p of 0 in $\mathbb{R}^k$ such that $\Phi_p : V_p \to M$ is an embedding. Note also that $\mathrm{Im}(\Phi_p)_{*0} = \mathrm{Im}\tilde{W}_p$.

Lemma 7.4. *There exists an open neighbourhood V_p of 0 in $\mathbb{R}^k$ such that $\mathrm{Im}(\Phi_p)_{*t} = \mathrm{Im}\tilde{W}_{\Phi_p(t)}$ for every $t = (t_1, t_2, ..., t_k) \in V_p$.*

Proof. We have

$$(\Phi_p)_{*t}(\frac{\partial}{\partial t_j}) = \big((\phi_{1,t_1})_* \circ ... \circ (\phi_{j-1,t_{j-1}})_* \circ X_{f_j} \circ \phi_{j+1,t_{j+1}} \circ ... \phi_{k,t_k}\big)(p)$$

$$= X_{h_j}(\Phi_p(t)) \in \mathrm{Im}\tilde{W}_{\Phi_p(t)},$$

where $h_j = f_j \circ (\phi_{1,t_1} \circ ... \circ \phi_{j-1,t_{j-1}})^{-1}$. Therefore, $\mathrm{Im}(\Phi_p)_{*t} \le \mathrm{Im}\tilde{W}_{\Phi_p(t)}$. However, $\dim \mathrm{Im}(\Phi_p)_{*t} = \dim \mathrm{Im}(\Phi_p)_{*0} = \dim \mathrm{Im}\tilde{W}_p = \dim \mathrm{Im}\tilde{W}_{\Phi_p(t)}$, since the flows of Hamiltonian vector fields consist of Poisson maps, for $t \in V_p$ such that $\Phi_p : V_p \to M$ is an embedding. □

If $q \in \Phi_p(V_p)$ and $\Phi_q : V_q \to M$ is an embedding constructed as Φ_p from functions g_1, g_2,...,$g_k \in C^\infty$, then there is an open neighbourhood V_0 of 0 in $\mathbb{R}^k$ such that Φ_q maps V_0 diffeomorphically onto an open subset of $\Phi_p(V_p)$, from the inverse function theorem.

Theorem 7.5. (Symplectic Stratification) *In a Poisson manifold M every symplectic leaf $S \subset M$ is an immersed submanifold and $T_pS = \mathrm{Im}\tilde{W}_p$ for every $p \in S$. Moreover, S has a unique symplectic structure such that S is a Poisson submanifold of M.*

Proof. Using the above notations, the family of all pairs $(\Phi_p(V_p), \Phi_p^{-1})$, $p \in S$, constructed from functions f_1, f_2,...,$f_k \in C^\infty(M)$ such that $\{X_{f_1}(p), X_{f_2}(p), ..., X_{f_k}(p)\}$ is a basis of $\mathrm{Im}\tilde{W}_p$, is a smooth atlas for S. Indeed, let p, $q \in S$ and $y \in \Phi_p(V_p) \cap \Phi_q(V_q)$. From the last remark, shrinking V_y we may assume that $\Phi_y(V_y) \subset \Phi_p(V_p) \cap \Phi_q(V_q)$ and Φ_y is an embedding of V_y similtaneously into $\Phi_p(V_p)$ and $\Phi_q(V_q)$. Therefore, S is an immersed Poisson submanifold of M and $T_pS = \mathrm{Im}\tilde{W}_p$, from Lemma 7.4. By Proposition 7.1, it remains to show that the structural matrix of S is invertible at every point $p \in S$. Let $f \in C^\infty(M)$ be such that $\{f, g\}(p) = 0$ for every $g \in C^\infty(M)$. Then $df(p)(X_g(p)) = X_g(f)(p) = 0$ for every $g \in C^\infty(M)$, which implies that $d(f|_S)(p) = df(p)|_{T_pS} = 0$. This shows that the structural matrix of S at p is invertible. □

Example 7.6. Let $(\mathfrak{g}, [,])$ be the Lie algebra of a Lie group G and $\mathfrak{g}^*$ be its dual. If $f \in C^\infty(\mathfrak{g}^*)$, the Hamiltonian vector field X_f with respect to the Poisson structure on $\mathfrak{g}^*$ defined in Example 7.2 satisfies

$$X_f(\mu)(g) = \{g, f\}(\mu) = \mu([dg(\mu), df(\mu)]) = -(\mu \circ \mathrm{ad}_{df(\mu)})(dg(\mu))$$

for every $g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, where we have identified $\mathfrak{g}^{**}$ with $\mathfrak{g}$. Thus, $X_f(\mu) = -(\mathrm{ad}_{df(\mu)})^*$ for every $\mu \in \mathfrak{g}^*$ and X_f is precisely a fundamental vector field of the coadjoint representation of G. It follows that the symplectic leaves in $\mathfrak{g}^*$ are the coadjoint orbits. Moreover, the restricted Poisson structure on each coadjoint orbit coincides with the Kirillov symplectic structure.

Chapter 3

Symmetries and integrability

3.1 Symplectic group actions

Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a smooth group action. If $X \in \mathfrak{g}$, the fundamental vector field $\phi_*(X) \in \mathcal{X}(M)$ of the action which corresponds to X is the infinitesimal generator of the flow $\phi_X : \mathbb{R} \times M \to M$ defined by $\phi_X(t,p) = \phi(\exp(tX), p)$. Note that for $g \in G$ the transformed vector field $(\phi_g)_*(\phi_*(X))$ is the fundamental vector field $\phi_*(\mathrm{Ad}_g(X))$, that is

$$(\phi_g)_{*p}(\phi_*(X)(p)) = \phi_*(\mathrm{Ad}_g(X))(\phi_g(p))$$

for every $p \in M$. Indeed,

$$\phi_*(\mathrm{Ad}_g(X))(\phi_g(p)) = \frac{d}{dt}\bigg|_{t=0} \phi^{\phi_g(p)}(\exp(t\mathrm{Ad}_g(X))) =$$

$$(\phi^{\phi_g(p)})_{*e}(\frac{d}{dt}\bigg|_{t=0} \exp(t\mathrm{Ad}_g(X))) = (\phi^{\phi_g(p)})_{*e}(\mathrm{Ad}_g(X)) =$$

$$\frac{d}{dt}\bigg|_{t=0} \phi(g\exp(tX)g^{-1}, \phi(g,p)) = \frac{d}{dt}\bigg|_{t=0} \phi(g\exp(tX), p) =$$

$$\frac{d}{dt}\bigg|_{t=0} (\phi^p \circ L_g)(\exp(tX)) = \frac{d}{dt}\bigg|_{t=0} (\phi_g \circ \phi^p)(\exp(tX)) =$$

$$(\phi_g)_{*p}((\phi^p)_{*e}(X)) = (\phi_g)_{*p}(\phi_*(X)(p)).$$

Lemma 1.1. *The linear map $\phi_* : \mathfrak{g} \to \mathcal{X}(M)$ is an anti-homomorphism of Lie algebras, meaning that $\phi_*([X,Y]) = -[\phi_*(X), \phi_*(Y)]$ for every X, $Y \in \mathfrak{g}$.*

Proof. If $p \in M$, then we compute

$$[\phi_*(X), \phi_*(Y)](p) = \frac{d}{dt}\bigg|_{t=0} (\phi_{\exp(-tX)})_{*\phi_{\exp(tX)}(p)}(\phi_*(Y))(\phi_{\exp(tX)}(p))) =$$

$$\frac{d}{dt}\bigg|_{t=0} \phi_*(\mathrm{Ad}_{\exp(-tX)}(Y))(p) = \phi_*(-\mathrm{ad}_X(Y))(p) = -\phi_*([X,Y]). \quad \square$$

Although ϕ_* is an anti-homomorphism of Lie algebras, it follows that $\phi_*(\mathfrak{g})$ is a Lie subalgebra of $\mathcal{X}(M)$ of finite dimension.

Definition 1.2. Let (M, ω) be a symplectic manifold and G a Lie group. A smooth group action $\phi : G \times M \to M$ is called *symplectic* if $\phi_g = \phi(g, .) : M \to M$ is a symplectomorphism for every $g \in G$.

If ϕ is symplectic, then $\phi_*(\mathfrak{g}) \subset \mathfrak{sp}(M, \omega)$, and therefore

$$\phi_*([\mathfrak{g}, \mathfrak{g}]) \subset [\mathfrak{sp}(M, \omega), \mathfrak{sp}(M, \omega)] \subset \mathfrak{h}(M, \omega),$$

by Proposition 4.8 in chapter 2. If $H_\phi : \mathfrak{g} \to H^1_{DR}(M)$ is the linear map defined by $H_\phi(X) = [i_{\phi_*(X)}\omega]$, then $X \in \text{Ker } H_\phi$ if and only if $\phi_*(X)$ is a Hamiltonian vector field, and $[\mathfrak{g}, \mathfrak{g}] \subset \text{Ker } H_\phi$.

Definition 1.3. A symplectic group action ϕ is called Hamiltonian if $H_\phi = 0$.

Thus, if $H^1_{DR}(M) = \{0\}$, then every symplectic group action on M is Hamiltonian. In particular, every symplectic group action on a simply connected symplectic manifold is Hamiltonian. Also if the Lie algebra $\mathfrak{g}$ of G is perfect, meaning that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then every symplectic group action of G is Hamiltonian. This happens for example in the case $G = SO(3, \mathbb{R})$, because $\mathfrak{so}(3, \mathbb{R})$ is isomorphic to the Lie algebra $(\mathbb{R}, \times)$, which is obviously perfect.

If ϕ is a Hamiltonian group action, in general there is no canonical way to choose a Hamiltonian function for $\phi_*(X)$, since adding a constant to a Hamiltonian function yields a new Hamiltonian function. If there is a linear map $\rho : \mathfrak{g} \to C^\infty(M)$ such that $\rho(X)$ is a Hamiltonian function for $\phi_*(X)$ for every $X \in \mathfrak{g}$, there is a smooth map $\mu : M \to \mathfrak{g}^*$ defined by $\mu(p)(X) = \rho(X)(p)$.

Examples 1.4. (a) Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ a smooth group action. Then, ϕ is covered by a group action $\tilde{\phi}$ of G on T^*M defined by $\tilde{\phi}(g, a) = a \circ (\phi_{g^{-1}})_{*\phi_g(\pi(a))}$, where $\pi : T^*M \to M$ is the cotangent bundle projection. Since $\pi \circ \tilde{\phi}_g = \phi_g \circ \pi$, differentiating we get

$$\pi_{*\tilde{\phi}_g(a)} \circ (\tilde{\phi}_g)_{*a} = (\phi_g)_{*\pi(a)} \circ \pi_{*a}$$

for every $a \in T^*M$ and $g \in G$. The Liouville 1-form θ on T^*M remains invariant under the action of G, because

$$((\tilde{\phi}_g)^*\theta)_a = \theta_{\tilde{\phi}_g(a)} \circ (\tilde{\phi}_g)_{*a} = a \circ (\phi_g^{-1})_{*\phi_g(\pi(a))} \circ (\phi_g)_{*\pi(a)} \circ \pi_{*a} = a \circ \pi_{*a} = \theta_a.$$

Consequently, the action of G on T^*M is symplectic with respect to the canonical symplectic structure $\omega = -d\theta$. Moreover, it is Hamiltonian, because

$$0 = L_{\tilde{\phi}_*(X)}\theta = i_{\tilde{\phi}_*(X)}(d\theta) + d(i_{\tilde{\phi}_*(X)}\theta)$$

and therefore $i_{\tilde{\phi}_*(X)}\omega = d(i_{\tilde{\phi}_*(X)}\theta)$. Here we have a linear map $\rho : \mathfrak{g} \to C^\infty(T^*M)$ defined by $\rho(X) = i_{\tilde{\phi}_*(X)}\theta$ and $\mu : T^*M \to \mathfrak{g}^*$ is given by the formula

$$\mu(a)(X) = \theta_a(\tilde{\phi}_*(X)).$$

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ be a coadjoint orbit. The symplectic Kirillov 2-form ω^- is Ad^*-invariant, by Lemma 5.6 of chapter 2, and so the natural action of G on $\mathcal{O}$ is symplectic. Recall that

$$\omega^-_\nu(X_{\mathfrak{g}^*}(\nu), Y_{\mathfrak{g}^*}(\nu)) = -\nu([X,Y]) = (\nu \circ \mathrm{ad}_Y)(X) = -Y_{\mathfrak{g}^*}(\nu)(X) = -X(Y_{\mathfrak{g}^*}(\nu))$$

for every X, $Y \in \mathfrak{g}$ and $\nu \in \mathcal{O}$, having identified $\mathfrak{g}^{**}$ with $\mathfrak{g}$. If now $\rho_X \in C^\infty(\mathfrak{g}^*)$ is the (linear) function defined by $\rho_X(\nu) = -\nu(X)$, then $d\rho_X(\nu) = -X$ (again we identify $\mathfrak{g}^{**}$ with $\mathfrak{g}$). It follows that $i_{X_{\mathfrak{g}^*}}\omega^- = d\rho_X$, which shows that the action of G on $\mathcal{O}$ is Hamiltonian.

Let $\phi : G \times M \to M$ be a Hamiltonian group action of the Lie group G with Lie algebra $\mathfrak{g}$ on a connected, symplectic manifold (M, ω). We assume that we have a linear lift $\rho : \mathfrak{g} \to C^\infty(M)$ such that $\phi_*(X) = X_{\rho(X)}$ for every $X \in \mathfrak{g}$. We shall study the possibility to change ρ to a new lift which is also a Lie algebra homomorphism. From Proposition 4.8 of chapter 2 and Lemma 1.1 we have

$$X_{\{\rho(X_0),\rho(X_1)\}} = -[X_{\rho(X_0)}, X_{\rho(X_1)}] = \phi_*([X_0, X_1]) = X_{\rho([X_0,X_1])},$$

for every X_0, $X_1 \in \mathfrak{g}$. Since M is connected, there exists $c(X_0, X_1) \in \mathbb{R}$ such that

$$\{\rho(X_0), \rho(X_1)\} = \rho([X_0, X_1]) + c(X_0, X_1).$$

Obviously, $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric, bilinear form. Moreover, $\delta c = 0$, from the Jacobi identity and the linearity of ρ. Hence $c \in Z^2(\mathfrak{g})$. If $\tilde{\rho} : \mathfrak{g} \to C^\infty(M)$ is another linear lift and $\sigma = \tilde{\rho} - \rho$, then $\sigma \in \mathfrak{g}^*$ and

$$\{\tilde{\rho}(X_0), \tilde{\rho}(X_1)\} = \{\rho(X_0), \rho(X_1)\} = \rho([X_0, X_1]) + c(X_0, X_1) =$$

$$\tilde{\rho}([X_0, X_1]) + c(X_0, X_1) - \sigma([X_0, X_1]).$$

Hence, $\tilde{c}(X_0, X_1) - c(X_0, X_1) = -\sigma([X_0, X_1]) = (\delta\sigma)(X_0, X_1)$. We conclude that there is a choice of $\tilde{\rho}$ such that $\tilde{c} = 0$ if and only if $[c] = 0$ in $H^2(\mathfrak{g})$. Thus, in case $H^2(\mathfrak{g}) = \{0\}$, we can always select a linear lift $\rho : \mathfrak{g} \to C^\infty(M)$ which is a Lie algebra homomorphism.

Examples 1.5. (a) Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ a smooth group action. As we saw in Example 1.4(a), the covering action $\tilde{\phi}$ on T^*M is Hamiltonian and $\rho : \mathfrak{g} \to C^\infty(T^*M)$ is given by the formula $\rho(X) = i_{\tilde{\phi}_*(X)}\theta$, where θ is the invariant Liouville 1-form. Then,

$$c(X_0, X_1) = -d\theta(\tilde{\phi}_*(X_0), \tilde{\phi}_*(X_1)) - \theta(\tilde{\phi}_*([X_0, X_1]) =$$

$$-L_{\tilde{\phi}_*(X_0)}\rho(X_1) + L_{\tilde{\phi}_*(X_1)}\rho(X_0) + \theta([\tilde{\phi}_*(X_0), \tilde{\phi}_*(X_1)]) - \theta(\tilde{\phi}_*([X_0, X_1]) =$$

$$-\{\rho(X_1), \rho(X_0)\} + \{\rho(X_0), \rho(X_1)\} - 2\theta(\tilde{\phi}_*([X_0, X_1]) = 2c(X_0, X_1)$$

and hence $c = 0$.

(b) If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ is a coadjoint orbit, then $\rho(X)(\nu) = -\nu(X)$ for every $X \in \mathfrak{g}$ and $\nu \in \mathcal{O} \subset \mathfrak{g}^*$, as we saw in Example 1.4(b). Therefore, $c = 0$, from the definition of the Kirillov 2-form.

(c) We shall now describe a simple example, where $[c]$ is a non-zero element of $H^2(\mathfrak{g})$. Let $G = (\mathbb{R}^2, +)$, in which case $\mathfrak{g} = \mathbb{R}^2$ with trivial Lie bracket. Let $M = \mathbb{R}^2$ endowed with the euclidean area 2-form $dx \wedge dy$. Let G act on M by translations. The action is symplectic and if $X = (a, b) \in \mathfrak{g}$, then

$$\phi_*(X) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

which is Hamiltonian with Hamiltonian function $\rho(X)(x, y) = ay - bx$. Then,

$$c((a_0, b_0), (a_1, b_1)) = a_0 b_1 - a_1 b_0$$

and therefore $[c] = c \neq 0$.

Definition 1.6. Let M be a symplectic manifold and G be a Lie group with Lie algebra $\mathfrak{g}$. A Hamiltonian group action $\phi : G \times M \to M$ is called *Poisson* (or *strongly Hamiltonian*) if there is a lift $\rho : \mathfrak{g} \to C^\infty(M)$ which is a Lie algebra homomorphism.

We conclude this section with a couple of criteria giving sufficient conditions for a symplectic group action to be Poisson.

Theorem 1.7. *Let (M, ω) be a compact, connected, symplectic $2n$-manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Then, every Hamiltonian group action $\phi : G \times M \to M$ is Poisson.*

Proof. Recall from Proposition 4.9 of chapter 2 that $C^\infty(M) = \mathbb{R} \oplus C_0^\infty(M, \omega)$. If $X \in \mathfrak{g}$ and $F \in C^\infty(M)$ is a Hamiltonian function of $\phi_*(X)$, we define

$$\rho(X) = F - \frac{1}{\mathrm{vol}(M)} \int_M F\omega^n,$$

where $\omega^n = \omega \wedge \omega \wedge ... \wedge \omega$ n-times. Then ρ is a linear lift. Let X_0, $X_1 \in \mathfrak{g}$ and F_0, $F_1 \in C^\infty(M)$ be Hamiltonian functions of $\phi_*(X_0)$ and $\phi_*(X_1)$, respectively. From Proposition 4.8 of chapter 2 and Lemma 1.1 we have

$$X_{\{F_0, F_1\}} = -[X_{F_0}, X_{F_1}] = -[\phi_*(X_0), \phi_*(X_1)] = \phi_*([X_0, X_1]),$$

and therefore

$$\rho([X_0, X_1]) = \{F_0, F_1\} - \frac{1}{\mathrm{vol}(M)} \int_M \{F_0, F_1\}\omega^n =$$

$$\{\rho(X_0), \rho(X_1)\} - 0 = \{\rho(X_0), \rho(X_1)\},$$

from Proposition 4.9(b) of chapter 2. $\square$

Theorem 1.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H^1(\mathfrak{g}) = \{0\}$ and $H^2(\mathfrak{g}) = \{0\}$, then every symplectic group action of G is Poisson.*

Proof. Let (M, ω) be a symplectic manifold and $\phi : G \times M \to M$ be a symplectic action of G. The condition $H^1(\mathfrak{g}) = \{0\}$ is equivalent to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. This implies that $\phi_*(\mathfrak{g}) \subset \mathfrak{h}(M, \omega)$, which means that the action is Hamiltonian. Since $H^2(\mathfrak{g}) = \{0\}$, the discussion preceeding the Examples 1.5 shows that the action is Poisson. $\square$

3.2 Momentum maps

Let (M, ω) be a connected, symplectic manifold, G be a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a Poisson action.

Definition 2.1. A *momentum map* for ϕ is a smooth map $\mu : M \to \mathfrak{g}^*$ such that $\rho : \mathfrak{g} \to C^\infty(M)$ defined by $\rho(X)(p) = \mu(p)(X)$ for $X \in \mathfrak{g}$ and $p \in M$ satisfies
(i) $\phi_*(X) = X_{\rho(X)}$, and
(ii) $\{\rho(X), \rho(Y)\} = \rho([X, Y])$ for every X, $Y \in \mathfrak{g}$.

From the point of view of dynamical systems, one reason to study momentum maps is the following. If $H : M \to \mathbb{R}$ is a G-invariant, smooth function, then μ is constant along the integral curves of the Hamiltonian vector field X_H. Indeed, for every $X \in \mathfrak{g}$ we have

$$L_{X_H}\rho(X) = \{\rho(X), H\} = -\{H, \rho(X)\} = -L_{\phi_*(X)}H = 0.$$

Theorem 2.2. *If G is a connected Lie group, then a momentum map $\mu : M \to \mathfrak{g}^*$ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.*

Proof. The momentum map μ is G-equivariant when $\mu(\phi_g(p)) = \mu(p) \circ \mathrm{Ad}_{g^{-1}}$ or equivalently

$$\rho(X)(\phi_g(p)) = \rho(\mathrm{Ad}_{g^{-1}}(X))(p)$$

for every $X \in \mathfrak{g}$, $g \in G$ and $p \in M$. Observe that if this is true for two elements g_1, $g_2 \in G$ and for every $X \in \mathfrak{g}$ and $p \in M$, then this is also true for the element $g_1 g_2$. Recall that since G is connected, if V is any connected, open neighbourhood of the identity $e \in G$ with $V = V^{-1}$, then $G = \bigcup_{n=1}^{\infty} V^n$, where $V^n = V \cdot \ldots \cdot V$, n-times. It follows that it suffices to prove the above equality for $g = \exp tY$ for $Y \in \mathfrak{g}$ and $t \in \mathbb{R}$. In other words, it suffices to show that

$$\rho(X)(\phi_{\exp(tY)}(p)) = \rho(\mathrm{Ad}_{\exp(-tY)}(X))(p)$$

for every X, $Y \in \mathfrak{g}$, $p \in M$ and $t \in \mathbb{R}$. As this is true for $t = 0$, we need only show that the two sides have equal derivatives with respect to t. The derivative of the left hand side is

$$\frac{d}{dt}\rho(X)(\phi_{\exp(tY)}(p)) = d\rho(X)(\phi_{\exp(tY)}(p))(\frac{d}{dt}\phi_{\exp(tY)}(p)) =$$

$$\omega(\phi_*(X)(\phi_{\exp(tY)}(p)), \phi_*(Y)(\phi_{\exp(tY)}(p))) =$$
$$\omega((\phi_g)_{*p}(\phi_*(\mathrm{Ad}_{\exp(-tY)}(X))(p)), (\phi_g)_{*p}(\phi_*(\mathrm{Ad}_{\exp(-tY)}(Y))(p))) =$$
$$\omega(\phi_*(\mathrm{Ad}_{\exp(-tY)}(X))(p), \phi_*(\mathrm{Ad}_{\exp(-tY)}(Y))(p)) =$$
$$\omega(\phi_*(\mathrm{Ad}_{\exp(-tY)}(X))(p), \phi_*(Y)(p)),$$

since $\mathrm{Ad}_{\exp(-tY)}(Y) = Y$, the action is symplectic and using the remarks in the beginning of section 1. The derivative of the right hand side is

$$\frac{d}{dt}\rho(\mathrm{Ad}_{\exp(-tY)}(X))(p) = \rho(\frac{d}{dt}\mathrm{Ad}_{\exp(-tY)}(X))(p) =$$
$$\rho(\mathrm{ad}_{(-Y)}(\mathrm{Ad}_{\exp(-tY)}(X)))(p) = \rho([-Y, \mathrm{Ad}_{\exp(-tY)}(X)])(p) =$$
$$\{\rho(\mathrm{Ad}_{\exp(-tY)}(X)), \rho(Y)\}(p) = \omega(\phi_*(\mathrm{Ad}_{\exp(-tY)}(X))(p), \phi_*(Y)(p)). \quad \Box$$

In general, for every $X \in \mathfrak{g}$ and $g \in G$ the smooth function

$$(\phi_g)^*(\rho(X)) - \rho(\mathrm{Ad}_{g^{-1}}(X)) : M \to \mathbb{R}$$

has differential

$$d((\phi_g)^*(\rho(X)) - \rho(\mathrm{Ad}_{g^{-1}}(X))) = (\phi_g)^*(d\rho(X)) - d\rho(\mathrm{Ad}_{g^{-1}}(X)) =$$
$$(\phi_g)^*(\tilde{\omega}^{-1}(\phi_*(X))) - \tilde{\omega}^{-1}(\phi_*(\mathrm{Ad}_{g^{-1}}(X))) = \tilde{\omega}^{-1}((\phi_{g^{-1}})_*\phi_*(X) - \phi_*(\mathrm{Ad}_{g^{-1}}(X))) = 0,$$

because the group action is symplectic and using the remarks in the beginning of section 1. Since M is connected, it is constant and so we have a function $c : G \to \mathfrak{g}^*$ defined by

$$c(g) = (\phi_g)^*(\rho(X)) - \rho(\mathrm{Ad}_{g^{-1}}(X)) = \mu(\phi_g(p)) - \mathrm{Ad}^*_g(\mu(p))$$

for any $p \in M$. If now g_0, $g_1 \in G$, then

$$c(g_0g_1) = \mu(\phi_{g_0}(\phi_{g_1}(p))) - \mathrm{Ad}^*_{g_0}(\mathrm{Ad}^*_{g_1}(\mu(p))) =$$
$$\mu(\phi_{g_0}(\phi_{g_1}(p))) - \mathrm{Ad}^*_{g_0}(\mu(\phi_{g_1}(p))) + \mathrm{Ad}^*_{g_0}(\mu(\phi_{g_1}(p))) - \mathrm{Ad}^*_{g_0}(\mathrm{Ad}^*_{g_1}(\mu(p))) =$$
$$c(g_0) + \mathrm{Ad}^*_{g_0}(\mu(\phi_{g_1}(p)) - \mathrm{Ad}^*_{g_1}(\mu(p))) = c(g_0) + \mathrm{Ad}^*_{g_0}(c(g_1)).$$

This means that c is a 1-cocycle with respect to the group cohomology of G^δ with coefficients in the G-module $\mathfrak{g}^*$, with respect to the coadjoint action, where G^δ denotes G made discrete. If μ' is another momentum map, there exists a constant $a \in \mathfrak{g}^*$ such that $\mu' = \mu + a$. The corresponding cocycle c' is given by the formula

$$c'(g) = \mu(\phi_g(p)) + a - \mathrm{Ad}^*_g(\mu(p)) - \mathrm{Ad}^*_g(a) = (c - \delta a)(g)$$

where δ denotes the coboundary operator in group cohomology. Thus, the cohomology class $[c] \in H^1(G^\delta; \mathfrak{g}^*)$ does not depend on the choice of the momentum map but only on the group action.

Proposition 2.3. *If $H^1(G^\delta;\mathfrak{g}^*)=\{0\}$, there exists a G-equivariant momentum map.*

Proof. Let μ be any momentum map with corresponding 1-cocycle c. There exists $a\in\mathfrak{g}^*$ such that $c=\delta a$, that is $c(g)=\mathrm{Ad}_g^*(a)-a$ for every $g\in G$. Then $\mu+a$ is a G-equivariant momentum map, because

$$\mu(\phi_g(p))+a=c(g)+\mathrm{Ad}_g^*(\mu(p))+a=\mathrm{Ad}_g^*(a)-a+\mathrm{Ad}_g^*(\mu(p))+a=\mathrm{Ad}_g^*(\mu+a)(p)$$

for every $g\in G$ and $p\in M$. □

Examples 2.4. (a) Let $\phi:G\times M\to M$ be a smooth action of the Lie group G with Lie algebra $\mathfrak{g}$ on the smooth manifold M and $\tilde\phi:G\times T^*M\to T^*M$ be the lifted action on the cotangent bundle. As we saw in Examples 1.4(a) and 1.5(a), the action of G on T^*M is Poisson and actually the Liouville 1-form θ on T^*M is G-invariant. The momentum map $\mu:T^*M\to\mathfrak{g}^*$ is given by the formula

$$\mu(a)(X)=\theta_a(\tilde\phi_*(X)(a))$$

for $X\in\mathfrak{g}$ and $a\in T^*M$, and is G-equivariant, because θ is G-invariant. Indeed,

$$\mu(\tilde\phi_g(a))(X)=\theta_{\tilde\phi(a)}(\tilde\phi_*(X)(\tilde\phi_g(a)))=((\tilde\phi_{g^{-1}})^*\theta)_{\tilde\phi(a)}(\tilde\phi_*(X)(\tilde\phi_g(a)))=$$

$$\theta_a((\tilde\phi_{g^{-1}})_{*\tilde\phi_g(a)}(\tilde\phi_*(X)(\tilde\phi_g(a))))=\theta_a(\tilde\phi_*(\mathrm{Ad}_{g^{-1}}(X))(a))=\mu(a)(\mathrm{Ad}_{g^{-1}}(X)),$$

for every $g\in G$.

In the case of the 3-dimensional euclidean space $\mathbb{R}^3$ we have $T^*\mathbb{R}^3\cong\mathbb{R}^3\times\mathbb{R}^3$, where the isomorphism is defined by the euclidean inner product $\langle,\rangle$, identifying thus $T^*\mathbb{R}^3$ with $T\mathbb{R}^3$. The Liouville 1-form is given by the formula

$$\theta_{(q,p)}(v,w)=\langle v,p\rangle.$$

The natural action of $SO(3,\mathbb{R})$ on $\mathbb{R}^3$ is covered by the action $\tilde\phi$ such that

$$\tilde\phi_A(q,p)(v)=\langle p,A^{-1}v\rangle=\langle Ap,v\rangle$$

for every $v\in T_q\mathbb{R}^3$ and $A\in SO(3,\mathbb{R})$. Therefore, $\tilde\phi_A(q,p)=(Aq,Ap)$ for every $(q,p)\in T^*\mathbb{R}^3$ and $A\in SO(3,\mathbb{R})$. If now $v\in\mathbb{R}^3\cong\mathfrak{so}(3,\mathbb{R})$, the corresponding fundamental vector field of the action satisfies

$$\tilde\phi_*(v)(q,p)=(\hat vq,\hat vp)=(v\times q,v\times p).$$

It follows that the momentum map satisfies

$$\mu(q,p)(v)=\langle v\times q,p\rangle=\langle q\times p,v\rangle$$

for every $v\in\mathbb{R}^3$. Consequently, the momentum map is the angular momentum

$$\mu(q,p)=q\times p.$$

Suppose now that we have a system of n particles in $\mathbb{R}^3$. The configuration space is $\mathbb{R}^{3n}$. The additive group $\mathbb{R}^3$ acts on $\mathbb{R}^{3n}$ by translations, that is

$$\phi_x(q^1, q^2, ..., q^n) = (q^1 + x, q^2 + x, ..., q^n + x)$$

for every $x \in \mathbb{R}^3$. The lifted action on $T^*\mathbb{R}^{3n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ is

$$\tilde{\phi}_x(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) = (q^1 - x, q^2 - x, ..., q^n - x, p_1, p_2, ..., p_n).$$

If now $X \in \mathbb{R}^3$, the corresponding fundamental vector field of the action is

$$\tilde{\phi}_*(X)(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) = (-X, -X, ..., -X, 0, 0, ..., 0).$$

Hence the momentum map $\mu : T^*\mathbb{R}^{3n} \to \mathbb{R}^3$ satisfies

$$\mu(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)(X) = \sum_{j=1}^{n} \langle -X, p_j \rangle = \langle X, -\sum_{j=1}^{n} p_j \rangle.$$

In other words, the momentum map in this case is the total linear momentum

$$\mu(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) = -\sum_{j=1}^{n} p_j.$$

This example justifies the use of the term momentum map.

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. As we saw in Examples 1.4(b) and 1.5(b), the transitive action of G on $\mathcal{O}$ is Poisson with momentum map $\mu : \mathcal{O} \to \mathfrak{g}^*$ given by the formula $\mu(\nu) = -\nu$ for every $\nu \in \mathcal{O}$. In other words, the momentum map is minus the inclusion of $\mathcal{O}$ in $\mathfrak{g}^*$, which is of course G-equivariant.

(c) Let h be the usual hermitian product and ω the standard symplectic 2-form in $\mathbb{C}^n$ defined by the formula $\omega(v, w) = \mathrm{Re}h(Jv, w)$, where $J : \mathbb{C}^n \to \mathbb{C}^n$ is multiplication by i. The natural group action $\phi : U(n) \times \mathbb{C}^n \to \mathbb{C}^n$ preserves h (by definition) and is symplectic, since the elements of $U(n)$ commute with J. If $X \in \mathfrak{u}(n)$, the corresponding fundamental vector field is $\phi_*(X)(z) = Xz$ and therefore

$$(i_{\phi_*(X)}\omega)_z(v) = \omega(Xz, v) = \mathrm{Re}h(JXz, v)$$

for every $v \in T_z\mathbb{C}^n$ and $z \in \mathbb{C}^n$. Let now $\rho(X) : \mathbb{C}^n \to \mathbb{R}$ be the smooth function defined by

$$\rho(X)(z) = \frac{i}{2}h(Xz, z)$$

which takes indeed real values since

$$h(Xz, z) = h(z, \bar{X}^t z) = h(z, -Xz) = -\overline{h(Xz, z)},$$

because $X \in \mathfrak{u}(n)$. Observe that

$$h(X(z+v), z+v) - h(Xz, z) = h(Xv, v) + h(Xz, v) - \overline{h(Xz, v)}$$

and

$$\lim_{v\to 0}\frac{h(Xv,v)}{\|v\|}=0.$$

It follows that

$$d\rho(X)(z)v=\frac{i}{2}[h(Xz,v)-\overline{h(Xz,v)}]=\mathrm{Re}(ih(Xz,v))=(i_{\phi_*(X)}\omega)_z(v)$$

for every $v\in T_z\mathbb{C}^n$ and $z\in\mathbb{C}^n$. This means that the action is Hamiltonian. Moreover, it is Poisson because for every X, $Y\in\mathfrak{u}(n)$ we have

$$\rho([X,Y])(z)=\rho(XY-YX)(z)=\frac{i}{2}[h(XYz,z)-h(YXz,z)]=$$

$$\frac{i}{2}[h(Yz,-Xz)-h(Xz,-Yz)]=\frac{i}{2}[-\overline{h(Xz,Yz)}+h(Xz,Yz)]=$$

$$\mathrm{Re}(ih(Xz,Yz))=\omega_z(Xz,Yz)=\{\rho(X),\rho(Y)\}(z).$$

In accordance to Theorem 2.2, the corresponding momentum map $\mu:\mathbb{C}^n\to\mathfrak{u}(n)^*$ is indeed $U(n)$-equivariant since

$$\mu(Az)(X)=\frac{i}{2}h(XAz,Az)=\frac{i}{2}h(\bar{A}^tXAz,z)=\frac{i}{2}h(A^{-1}XAz,z)=$$

$$\frac{i}{2}h(\mathrm{Ad}_{A^{-1}}(X)z,z)=(\mu(z)\circ\mathrm{Ad}_{A^{-1}})(X)$$

for every $z\in\mathbb{C}^n$, $A\in U(n)$ and $X\in\mathfrak{u}(n)$, because $\mathrm{Ad}_{A^{-1}}(X)=A^{-1}XA$.

3.3 Symplectic reduction

Let (M,ω) be a connected, symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\phi:G\times M\to M$ a symplectic action. In general, the orbit space $G\backslash M$ of the action may not be a smooth manifold (not even a Hausdorff space). Even in the case it is, it may not admit any symplectic structure, as for instance it may be odd dimensional. If the action is Poisson and there is a G-equivariant momentum map $\mu:M\to\mathfrak{g}^*$, there exists a well defined continuous map $\tilde{\mu}:G\backslash M\to G\backslash\mathfrak{g}^*$. Under certain circumstances, the level sets $\tilde{\mu}^{-1}(\mathcal{O}_a)$, $a\in\mathfrak{g}^*$, can be given a symplectic structure in a natural way. It is easy to see that the inclusion $j:\mu^{-1}(a)\hookrightarrow\mu^{-1}(\mathcal{O}_a)$ induces a continuous bijection $j_\#:G_a\backslash\mu^{-1}(a)\to G\backslash\mu^{-1}(\mathcal{O}_a)$. In certain cases, $j_\#$ is a homeomorphism or even a diffeomorphism of smooth manifolds. For example, if the action of G on M is free and proper and a is a regular value of μ, then $\mu^{-1}(\mathcal{O}_a)$ is a smooth submanifold of M and so are $G\backslash\mu^{-1}(\mathcal{O}_a)$ and $G_a\backslash\mu^{-1}(a)$. Moreover, in this case $j_\#$ is a diffeomorphism. In particular, these are true if G is compact and the action is free.

Definition 3.1. Let P, Q be two smooth manifolds and $f:P\to Q$ be a smooth map. A point $q\in Q$ is called a *clean* (or *weakly regular*) value of f if $f^{-1}(q)$ is an embedded smooth submanifold of M and $T_pf^{-1}(q)=\mathrm{Ker}f_{*p}$ for every $p\in f^{-1}(q)$.

Obviously, a regular value is always clean, but the converse is not true. For example, $(0,0) \in \mathbb{R}^2$ is a clean, but not regular, value of the smooth function $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x,y,z) = (z^2, z)$.

Theorem 3.2. *Let (M,ω) be a symplectic manifold, G be a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ be a Poisson action with a G-equivariant momentum map $\mu : M \to \mathfrak{g}^*$. Let $a \in \mathfrak{g}^*$ be a clean value of μ such that the orbit space $M_a = G_a \backslash \mu^{-1}(a)$ is a smooth manifold and the quotient map $\pi_a : \mu^{-1}(a) \to M_a$ is a smooth submersion, where G_a is the isotropy group of a with respect to the coadjoint action. Then there exists a unique symplectic 2-form ω_a on M_a auch that $\pi_a^*\omega_a = \omega|_{\mu^{-1}(a)}$.*

Proof. First note that $\mu^{-1}(a)$ is indeed G_a-invariant, since μ is G-equivariant. Evidently, $\tilde{\omega}_a = \omega|_{\mu^{-1}(a)}$ is closed and G_a-invariant, because the action is symplectic. So there exists a unique 2-form ω_a on M_a such that $\pi_a^*\omega_a = \tilde{\omega}_a$. Since π_a is a submersion and $\tilde{\omega}_a$ is closed, so is ω_a. It remains to show that ω_a is non-degenerate.

Observe that for any $p \in \mu^{-1}(a)$ we have

$$(T_pGp)^\perp = \{v \in T_pM : \omega_p(\phi_*(X)(p), v) = 0 \text{ for every } X \in \mathfrak{g}\} =$$

$$\{v \in T_pM : \mu_{*p}(v)(X) = 0 \text{ for every } X \in \mathfrak{g}\} =$$

$$\mathrm{Ker}\mu_{*p} = T_p\mu^{-1}(a),$$

because T_pGp is generated by the values at p of the fundamental vector fields of the action. On the other hand, $\mu^{-1}(a) \cap Gp = G_ap$, since μ is G-equivariant, and therefore $T_pG_ap \subset T_p\mu^{-1}(a) \cap T_pGp$. Actually, we have equality. To see this, let $v \in T_p\mu^{-1}(a) \cap T_pGp$. There exists $X \in \mathfrak{g}^*$ such that $v = \phi_*(X)(p)$ and since $T_p\mu^{-1}(q) = \mathrm{Ker}\mu_{*p}$, we have

$$0 = \mu_{*p}(v)(X) = \left.\frac{d}{dt}\right|_{t=0} \mu(\phi_{\exp(tX)}(p)) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}(\mu(p)) = (\mathrm{Ad}^*)_*(X)(a).$$

This means $X \in \mathfrak{g}_a$, the Lie algebra of G_a, or in other words $v \in T_pG_ap$.

It follows that $T_pG_ap = T_p\mu^{-1}(a) \cap (T_p\mu^{-1}(a))^\perp$. Suppose now that $v \in T_p\mu^{-1}(a)$ is such that $\tilde{\omega}_a(v,w) = 0$ for every $w \in T_p\mu^{-1}(a)$. Then $v \in (T_p\mu^{-1}(a))^\perp$ and so $v \in T_pG_ap$. Hence $(\pi_a)_{*p}(v) = 0$. This proves that ω_a is non-degenerate. □

Under the assumptions of Theorem 3.2 let $H \in C^\infty(M)$ be G-invariant. As we observed in the beginning of section 3.2, the momentum map μ is constant along the integral curves of the Hamiltonian vector field X_H, which is obviously G-invariant, since the action is symplectic. Thus, X_H is tangent to $\mu^{-1}(a)$ and is G_a-invariant. Let $H_a \in C^\infty(M_a)$ be defined by $H_a \circ \pi_a = H$ and X_{H_a} be the corresponding Hamiltonian vector field on the symplectic manifold (M_a, ω_a). Then $(\pi_a)_*X_H = X_{H_a}$, because for every $p \in \mu^{-1}(a)$ and $v \in T_p\mu^{-1}(a)$ we have

$$(\omega_a)_{\pi_a(p)}((\pi_a)_{*p}(X_H(p)), (\pi_a)_{*p}(v)) = ((\pi_a)^*\omega_a)_p(X_H(p), v) = \omega_p(X_H(p), v) =$$

$$dH(p)(v) = dH_a(\pi_a(p))((\pi_a)_{*p}(v)) = (\omega_a)_{\pi_a(p)}(X_{H_a}(\pi_a(p)), (\pi_a)_{*p}(v)).$$

The Hamiltonian vector field X_{H_a} is called the *reduced Hamiltonian vector field.* This is a geometric way to use the symmetry group G of X_H in order to reduce the number of differential equations we have to solve, if we want to find its integral curves.

Examples 3.3. (a) Let M be a symplectic manifold and $H \in C^\infty(M)$ be such that the Hamiltonian vector field X_H is complete. Its flow is a Poisson group action of $\mathbb{R}$ on M with momentum map H itself. Since $\mathbb{R}$ is abelian, the coadjoint action is trivial. If now $a \in \mathbb{R}$ is a clean value of H, then according to Theorem 3.2 the orbit space $\mathbb{R}\backslash H^{-1}(a)$ has a natural symplectic structure.

(b) Let $SO(3,\mathbb{R})$ act on $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ as in the Example 2.4(a). As we saw, the momentum map $\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the angular momentum

$$\mu(q,p) = q \times p.$$

The Jacobian matrix of μ at (q,p) is $(-\hat{p},\hat{q})$, and so every non-zero $v \in \mathbb{R}^3$ is a regular value of μ. As the Example 2.5.2 shows, the isotropy group of v is the group of rotations of $\mathbb{R}^3$ around the axis generated by v, hence isomorphic to S^1. Thus, the orbit space $S^1\backslash\mu^{-1}(v)$ has a symplectic structure.

(c) Let $\phi : S^1 \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the action with $\phi(e^{it},z) = e^{it}z$. As we saw in the Example 2.4(c), the action is Poisson with respect to the standard symplectic structure of $\mathbb{C}^{n+1}$ and the momentum map $\mu : \mathbb{C}^{n+1} \to \mathfrak{u}(1)^* = (i\mathbb{R})^* \cong \mathbb{R}$ is given by the formula

$$\mu(z) = \frac{i}{2}h(iz,z) = -\frac{1}{2}|z|^2.$$

Now $a = -\dfrac{1}{2}$ is a regular value of μ and $\mu^{-1}(a) = S^{2n+1}$. Since S^1 is abelian, we conclude that $\mathbb{C}P^n = S^1\backslash S^{2n+1}$ has a symplectic 2-form. It is clear from the definitions that this is exactly the symplectic 2-form that was described in the Example 2.2.3.

(d) Let M be a symplectic $2n$-manifold and H_1,...,$H_k \in C^\infty(M)$ such that the Hamiltonian vector fields X_{H_1},...,X_{H_k} are complete. If $\{H_i,H_j\} = 0$ for every i, $j = 1,2,...,k$, then their flows commute and define a Poisson action of $\mathbb{R}^k$ on M with momentum map $\mu = (H_1,...,H_k) : M \to \mathbb{R}^k$. Since $\mathbb{R}^k$ is abelian, we get a symplectic structure on the orbit space $\mathbb{R}^k\backslash\mu^{-1}(a)$ for every clean value $a \in \mathbb{R}^k$ of μ. In the next section we shall examine this situation in further detail when $k = n$.

3.4 Completely integrable Hamiltonian systems

Let (M,ω) be a connected, symplectic $2n$-manifold and $H_1 \in C^\infty(M)$. The triple (H_1,M,ω) is called a *completely integrable Hamiltonian system* if there are H_2,...,$H_n \in C^\infty(M)$ such that $\{H_i,H_j\} = 0$ for every $1 \le i,j \le n$ and the differential 1-forms dH_1, dH_2,...,dH_n are linearly independent on a dense open set $D \subset M$. In this section we shall always assume that we have such a system.

For every $p \in M$ the set $\{X_{H_1}(p), X_{H_2}(p), ..., X_{H_n}(p)\}$ generates an isotropic linear subspace of T_pM. If $p \in D$, then it is a basis of a Lagrangian subspace of T_pM. If $f = (H_1,H_2,...,H_n) : M \to \mathbb{R}^n$, then $f|_D$ is a smooth submersion and so the connected components of the fibers $f^{-1}(y) \cap D$, $y \in \mathbb{R}^n$, are the leaves of a foliation of D by Lagrangian submanifolds, because $f_{*p}(X_{H_i}(p)) = 0$ for every $1 \leq i \leq n$.

Suppose that the Hamiltonian vector fields X_{H_1}, X_{H_2},..., X_{H_n} are complete. Since their flows commute, they define a Poisson group action $\phi : \mathbb{R}^n \times M \to M$ with fundamental vector fields X_{H_1}, X_{H_2},..., X_{H_n} and momentum map f. Let $y \in \mathbb{R}^n$ be a regular value of f. Then $f^{-1}(y) \subset D$ is a $\mathbb{R}^n$-invariant, regular n-dimensional submanifold of M. The vector fields X_{H_1}, X_{H_2},..., X_{H_n} are tangent to $f^{-1}(y)$, and since they are linearly independent at every point of $f^{-1}(y)$, every orbit in $f^{-1}(y)$ is an open subset of $f^{-1}(y)$. This implies that every connected component N of $f^{-1}(y)$ is an orbit of the action. Thus, N is diffeomorphic to the homogenenous space $\mathbb{R}^n/\Gamma_p$, where Γ_p is the isotropy group of p. Note that Γ_p does not depend on p, but only on N, since $\mathbb{R}^n$ is abelian. Also, Γ_p is a 0-dimensional closed subgroup of $\mathbb{R}^n$ and therefore is discrete. The discrete subgroups of $\mathbb{R}^n$ are described as follows.

Lemma 4.1. *Let $\Gamma \leq \mathbb{R}^n$ be a non-trivial discrete subgroup. Then Γ is a lattice, that is there exist $1 \leq k \leq n$ and linearly independent vectors v_1,...,v_k such that*

$$\Gamma = \mathbb{Z}v_1 + ... + \mathbb{Z}v_k.$$

Proof. Let $u_1 \in \Gamma \setminus \{0\}$. Since Γ is discrete, there exists $\lambda > 0$ such that $\lambda u_1 \in \Gamma$ and $\Gamma \cap (-\lambda, \lambda)u_1 = \{0\}$. Let $v_1 = \lambda u_1$ and so $\Gamma \cap \mathbb{R}v_1 = \mathbb{Z}v_1$. If $\Gamma = \mathbb{Z}v_1$, then $k = 1$ and we have finished. Suppose that $\Gamma \neq \mathbb{Z}v_1$ and $u_2 \in \Gamma \setminus \mathbb{Z}v_1$ be such that $\Gamma \cap \mathbb{R}u_2 = \mathbb{Z}u_2$. Then, v_1 and u_2 are linearly independent. Let

$$P(v_1, u_2) = \{t_1 v_1 + t_2 u_2 : t_1, t_2 \in [0, 1]\}$$

be the parallelogram generated by v_1 and u_2. The set $\Gamma \cap P(v_1, u_2)$ is finite, because Γ is discrete. So there exists $v_2 \in P(v_1, u_2)$ such that $\Gamma \cap P(v_1, v_2) = \{0, v_1, v_2, v_1 + v_2\}$. It follows now that

$$\Gamma \cap (\mathbb{R}v_1 \oplus \mathbb{R}v_2) = \mathbb{Z}v_1 + \mathbb{Z}v_2,$$

because if there exist t_1, $t_2 \in \mathbb{R} \setminus \mathbb{Z}$ such that $t_1 v_1 + t_2 v_2 \in \Gamma$, then

$$(t_1 - [t_1])v_1 + (t_2 - [t_2])v_2 \in \Gamma \cap P(v_1, v_2),$$

contradiction. If $\Gamma = \mathbb{Z}v_1 + \mathbb{Z}v_2$, then $k = 2$ and we have finished. If not, then we proceed inductively using the same argument repeatedly, replacing the parallelograms with parallelopipeds etc. Since $\mathbb{R}^n$ has finite dimension, we end up with linearly independent vectors v_1,...,v_k such that $\Gamma = \mathbb{Z}v_1 + ... + \mathbb{Z}v_k$. $\square$

Corollary 4.2. *Let $\Gamma \leq \mathbb{R}^n$ be a non-trivial discrete subgroup. Then there exists $1 \leq k \leq n$ such that the homogenenous space $\mathbb{R}^n/\Gamma$ is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$. If $\mathbb{R}^n/\Gamma$ is compact, then $k = n$ and $\mathbb{R}^n/\Gamma$ is diffeomorphic to the n-torus T^n.*

Proof. From Lemma 4.1 there exist $1 \leq k \leq n$ and linearly independent vectors v_1,...,v_k such that $\Gamma = \mathbb{Z}v_1 + ... + \mathbb{Z}v_k$. We complete to a basis $\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ of $\mathbb{R}^n$ and consider the linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(v_j) = e_j$, $1 \leq j \leq n$.

Then, $T(\Gamma) = \mathbb{Z}^k \times \{0\}$, and so T imduces a diffeomorphism $\tilde{T} : \mathbb{R}/\Gamma \to T^k \times \mathbb{R}^{n-k}$. The rest is obvious. $\square$

Note that if N is compact then the restrictions of the Hamiltonian vector fields X_{H_1}, X_{H_2},..., X_{H_n} to N are automatically complete. So, we have arrived at the following.

Theorem 4.3. (Arnold-Liouville) *Let $y \in \mathbb{R}^n$ be a regular value of f and N be a connected component of $f^{-1}(y)$.*

(i) If N is compact, then it is diffeomorphic to the n-torus T^n.

(ii) If N is not compact and X_{H_1}, X_{H_2},..., X_{H_n} are complete, then N is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$. $\square$

It is not hard now to describe the flow of the Hamiltonian vector field X_{H_1} on N. Let $p \in N$ and $\tilde{\phi^p} : \mathbb{R}^n/\Gamma \to N$ be the diffeomorphism which is induced by $\phi^p = \phi(.,p) : \mathbb{R}^n \to N$. Let $(\psi_t)_{t\in\mathbb{R}}$ be the flow of X_{H_1} on N and $\tilde{\psi_t} = (\tilde{\phi^p})^{-1} \circ \psi_t \circ \tilde{\phi^p}$, $t \in \mathbb{R}$, be the conjugate flow on $\mathbb{R}^n/\Gamma$. Then

$$\tilde{\psi_t}([t_1, ..., t_n]) = (\tilde{\phi^p})^{-1}(\psi_t(\phi((t_1, ..., t_n), p))) =$$

$$(\tilde{\phi^p})^{-1}(\phi((t + t_1, t_2, ..., t_n), p)) = [t + t_1, t_2, ..., t_n].$$

In other words, $\psi_t([v]) = [v + te_1]$ for every $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Using the notations of the proof of Corollary 4.2, let $T(e_1) = (\nu_1, ..., \nu_n)$ and let $\chi_t = \tilde{T} \circ \tilde{\psi_t} \circ \tilde{T}^{-1}$, $t \in \mathbb{R}$, be the conjugate flow on $T^k \times \mathbb{R}^{n-k}$. Then,

$$\chi_t(e^{2\pi i t_1}, ..., e^{2\pi i t_k}, t_{k+1}, ..., t_n) = \tilde{T}(\tilde{\psi_t}([t_1 v_1 + ... + t_n v_n])) = \tilde{T}([te_1 + t_1 v_1 + ... + t_n v_n]).$$

Since

$$T(te_1 + t_1 v_1 + ... + t_n v_n) = tT(e_1) + t_1 e_1 + ... + t_n e_n = (t_1 + t\nu_1, ..., t_n + t\nu_n)$$

it follows that

$$\chi_t(e^{2\pi i t_1}, ..., e^{2\pi i t_k}, t_{k+1}, ..., t_n) = (e^{2\pi i(t_1+t\nu_1)}, ..., e^{2\pi i(t_k+t\nu_k)}, t_{k+1}+t\nu_{k+1}, ..., t_n+t\nu_n).$$

This shows that the flow of X_{H_1} on N is smoothly conjugate to a linear flow on $T^k \times \mathbb{R}^{n-k}$. In case N is compact, then $k = n$ and the real numbers ν_1,...,ν_n are called the *frequences of the flow* on N. As is well known, if they are linearly independent over $\mathbb{Q}$, then the flow on N is uniquely ergodic and every orbit is dense in N.

In the rest of this section we shall study more closely the case of a compact connected component N of $f^{-1}(y)$, where $y \in \mathbb{R}^n$ is a regular value of f. We are mainly interested in the structure of (M, ω) around N. Since N is compact, there exist an open neighbourhood U of y and a ϕ-invariant neighbourhood V of N such that $\overline{V}$ is compact, $f(V) = U$ and $f|_V : V \to U$ is a submersion with compact fibers. Therefore, $f|_V$ is a locally trivial fibration with Lagrangian fibers diffeomorphic to the n-torus T^n. Shrinking U to an open neighbourhood of y diffeomorphic to $\mathbb{R}^n$, we get an open neighbourhood V of N diffeomorphic to $U \times N$ and so to $\mathbb{R}^n \times T^n$.

The orbits of the restriction of the Poisson group action ϕ on V are the fibers $(f|_V)^{-1}(q)$, $q \in U$, and the isotropy group of a point on $(f|_V)^{-1}(q)$ depends only on q, since $\mathbb{R}^n$ is abelian. We shall show first that the isotropy groups vary smoothly with q. Let $t_0 \in \Gamma_p \setminus \{0\}$, where $p \in N$, and let $s : U \to V$ be a smooth section, that is $f \circ s = id$. Identifying a small open neighbourhood B of p in N with $\mathbb{R}^n$, there exist an open neighbourhood W of t_0 in $\Gamma_p \setminus \{0\}$ and an open neighbourhood U_y of y in U such that

$$pr(\phi(t, s(q))) - s(q) \in B$$

for every $t \in W$ and $q \in U_y$, where $pr : V \to N$ is the projection. The smooth map $G : W \times U_y \to B$ with

$$G(t, q) = pr(\phi(t, s(q))) - s(q)$$

is thus well defined and 0 is a regular value of $G(., y)$. From the Implicit Function Theorem there exist an open neighbourhood U'_y of y and a smooth map $h : U'_y \to \mathbb{R}^n$ such that $G(h(q), q) = 0$ for every $q \in U'_y$ and $h(y) = t_0$. In other words,

$$\phi(h(q), s(q)) = s(q)$$

for every $q \in U'_y$. Varying now t_0 in a basis of the lattice Γ_p, we conclude that there exists smooth functions v_1,...,$v_n : U \to \mathbb{R}^n$ such that

$$\Gamma_p = \mathbb{Z}v_1(f(p)) + ... + \mathbb{Z}v_n(f(p))$$

for every $p \in V$, shrinking U and V appropriately.

Let now Y_i be the infinitesimal generator of the smooth flow $\phi^i : \mathbb{R} \times V \to V$ with $\phi^i(t, p) = \phi(tv_i(f(p)), p)$. Then,

$$Y_i(p) = \sum_{j=1}^{n} v_{i,j}(f(p))X_{H_j}(p),$$

where $v_i = (v_{i,1}, ..., v_{i,n})$. Obviously, the flow ϕ^i is periodic with period 1, the vector fields Y_1,...,Y_1 are linearly independent and $[Y_i, Y_j] = 0$, because $[X_{H_i}, X_{H_j}] = 0$ and X_{H_1},...,X_{H_n} are tangent to the fibers of $f|_V$. This means that there is a well defined group action of the n-torus T^n on V with fundamental vector fields Y_1,...,Y_n, whose orbits are the fibers of $f|_V$. We shall show that this action is Poisson and and we shall construct a momentum map. First note that since we have selected U to be contractible and V is diffeomorphic to $U \times N$, the inclusion $N \subset V$ induces an isomorphism in cohomology. It follows that $\omega|_V$ is exact since $\omega|_N = 0$, because N is Lagrangian. Let η be a smooth 1-form on V such that $\omega|_V = -d\eta$ and for $1 \leq i \leq n$ let $g_i : V \to \mathbb{R}$ be the smooth function defined by

$$g_i(p) = \int_0^1 (i_{Y_i}\eta)(\phi^i(t, p))dt.$$

Since T^n is abelian, it suffices to show that $Y_i = X_{g_i}$ for all $1 \leq i \leq n$, that is

$$\omega(Y_i(p), Z(p)) = dg_i(p)(Z(p))$$

for every $Z(p) \in T_pM$ and $p \in V$. Then, $(g_1, ..., g_n)$ will be a momentum map.

Let $Z(p) \in T_pM$ and let Z be an extension to a smooth vector field on V which is invariant by the action of T^n, that is $[Y_i, Z] = 0$ for every $1 \leq i \leq n$. Differentiating g_i we get

$$dg_i(p)(Z(p)) = \int_0^1 d(i_{Y_i}\eta)(\phi^i(t,p))(Z(\phi^i(t,p)))dt,$$

since Z is ϕ^i-invariant. But

$$d(i_{Y_i}\eta)(Z) = L_Z(i_{Y_i}\eta) = i_{Y_i}(L_Z\eta) + \eta([Z, Y_i]) = (L_Z\eta)(Y_i) =$$

$$d(i_Z\eta)(Y_i) + i_Z(d\eta)(Y_i) = d(i_Z\eta)(Y_i) + \omega(Y_i, Z)$$

and

$$\int_0^1 d(i_Z\eta)(Y_i)(\phi^i(t,p))dt = (i_z\eta)(\phi^i(1,p)) - (i_z\eta)(\phi^i(0,p)) = 0,$$

since the flow ϕ^i is periodic with period 1. Consequently,

$$dg_i(p)(Z(p)) = \int_0^1 \omega(Y_i, Z)(\phi^i(t,p))dt$$

and it suffices to show that $\omega(Y_i, Z)$ is ϕ^i-invariant. Indeed, since $[Y_i, Z] = 0$, we have $L_{Y_i}(\omega(Y_i, Z)) = i_{Y_i}(L_{Y_i}\omega)(Z)$ and

$$i_{Y_i}(L_{Y_i}\omega) = i_{Y_i}(d(i_{Y_i}\omega)) = \sum_{j=1}^n Y_i(v_{i,j} \circ f)dH_j - \sum_{j=1}^n Y_i(H_j)d(v_{i,j} \circ f) = 0,$$

because $Y_i(v_{i,j} \circ f) = Y_i(H_j) = 0$, since H_j and $v_{i,j} \circ f$, $1 \leq j \leq n$, are constant along the orbits of Y_i.

Since now $(g_1, ..., g_n)$ is a momentum map of the action of T^n, it is constant on the fibers of $f|_V$ and so it is a function of $f(p)$, $p \in V$. We shall henceforth consider $(g_1, ..., g_n)$ as a function defined on U. Its rank at every point is n because

$$dg_1 \wedge ... \wedge dg_n = \frac{1}{n!} i_{Y_1}...i_{Y_n}(\omega \wedge ... \wedge \omega).$$

Considering local coordinates θ_1,...θ_n on N around a point $p \in N$, the smooth map $g = (g_1, ..., g_n, \theta_1, ..., \theta_n) : V \to \mathbb{R}^{2n}$ defines local coordinates in a small neighbourhood of p in M. Moreover, since $(g_1, ..., g_n)$ is a momentum map of the n-torus action and the action on the fibers is simply translation, we have

$$(g^{-1})^*\omega = \sum_{i=1}^n d\theta_i \wedge dg_i + \sum_{i<j} a_{ij} dg_i \wedge dg_j$$

for some smooth functions a_{ij}, $1 \leq i < j \leq n$. The fact that ω is closed implies that the 2-form

$$\alpha = \sum_{i<j} a_{ij} dg_i \wedge dg_j$$

is also closed, which means that a_{ij} does not depend on θ_1,...,θ_n. Having chosen U contractible, there exists a smooth 1-form β such that $\alpha = -d\beta$. So, there are smooth functions β_1,...,β_n of g_1,..,g_n such that

$$\beta = \sum_{i=1}^{n} \beta_i dg_i.$$

Putting now $\psi_i = \theta_i - \beta_i$ we get

$$(g^{-1})^*\omega = \sum_{i=1}^{n} d\psi_i \wedge dg_i.$$

The local coordinates $(g_1, ..., g_n, \psi_1, ..., \psi_n)$ are called *action angle coordinates.*